

Non-smooth saddle-node bifurcations III: strange attractors in continuous time

G. Fuhrmann[†]

31st December 2015

Non-smooth saddle-node bifurcations give rise to minimal sets of interesting geometry built of so-called strange non-chaotic attractors. We show that certain families of quasiperiodically driven logistic differential equations undergo a non-smooth bifurcation. By a previous result on the occurrence of non-smooth bifurcations in forced discrete time dynamical systems, this yields that within the class of families of quasiperiodically driven differential equations, non-smooth saddle-node bifurcations occur in a set with non-empty C^2 -interior.

1 Introduction

The logistic differential equation is a model for single-species populations and as such it is certainly among the most famous ode's from mathematical biology. Much research has been carried out in order to understand the behaviour of a single species in a fluctuating environment both in the applied sciences as well as in pure math [4, 5, 8–11, 18, 27, 34, 37, 38, 43, 46]. Having in mind that tidal effects—which result from the gravitational interplay of the moon and the sun—are almost surely quasiperiodic, it is particularly desirable to understand the effect of quasiperiodic forcing when seeking an understanding of long-term ecological behaviour (see, e.g. [39, 40]). However, up to now, there are few studies taking into account quasiperiodic forcing of the logistic equation in continuous time (see the discussion below Theorem 2.2).

An intriguing feature of quasiperiodically forced systems is the occurrence of strange non-chaotic attractors (SNA's). In the discrete time setting, the mechanism behind the creation of SNA's as well as their geometry are well-understood [14, 15, 17, 24–26, 31]. In [14], the author has shown that within the class of C^2 -families of quasiperiodically forced (qpf) monotone interval maps, SNA's occur in a set with non-empty interior (see Theorem 1.11 below). While the occurrence of strange attractors in systems with low dynamical complexity (in the sense of zero entropy) is with no doubt a fascinating fact in itself, it seems worth remembering that a main motivation for their study actually came from ode's driven at incommensurate frequencies [16]. As a matter of fact, the first examples of SNA's happened to be encountered in a continuous time setting [32, 33, 35, 44] (for a discussion, see also [28, 29]). Nonetheless, all examples in the continuous time setting are basically projective

[†]Institute of Mathematics, FSU Jena, Germany. Email: gabriel.fuhrmann@uni-jena.de

actions of linear cocycles [7, 32, 33, 35, 44].¹

A natural setting for the creation of SNA's are non-smooth saddle-node bifurcations of one-parameter families of driven one-dimensional systems (see Section 1.2). Here, they occur as the outcome of the collision of two continuous invariant curves. The present work shows that the property of undergoing such a non-smooth bifurcation has—analogously to the discrete time case—non-empty interior in the C^2 -topology in the class of qpf families of one-dimensional ode's (see Theorem 2.2).

The proof of this—in a sense—abstract fact has a consequence which is important from the applied point of view introduced above: our core idea is to consider the logistic differential equation with quasiperiodic additive forcing and reduce its dynamics—by means of a suitable Poincaré section—to those of qpf maps that verify the assumptions of Theorem 1.11, that is, to maps for which there exists an SNA. Now, an easy argument shows that the respective reduced system possesses an SNA if and only if the original system does (see Section 3). While the main work thus happens to be the rather technical analysis of the qpf logistic ode (carried out in Section 4), the robustness of non-smooth bifurcations in the continuous time case comes as a by-product of the application of Theorem 1.11. Furthermore, with little extra effort, we can carry over the geometric findings from the discrete time setting in [15] to the present situation (see Theorem 2.3).

Our main results are contained in Section 2. Their proofs can be found in the last section of this article. In the remainder of the current section, we introduce some basic notation and review some facts and definitions from non-autonomous bifurcation theory, fractal geometry and the discrete time analogue to what we consider in this work.

Acknowledgements. I would like to thank Tobias Jäger for introducing me to the problem and for his helpful remarks on an earlier version of this manuscript. This work was supported by an Emmy-Noether-Grant of the German Research Council (DFG grant JA 1721/2-1) and is part of the activities of the Scientific Network “Skew product dynamics and multifractal analysis” (DFG grant OE 538/3-1).

1.1 Setting and Notation

Throughout this article, let $X \subseteq \mathbb{R}$ be a non-degenerate interval (possibly non-compact), $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $D \geq 2$. Given a *non-autonomous vector field*, that is, a map $F: \mathbb{T}^D \times X \rightarrow \mathbb{R}$, we study (*local*) *skew product flows* or, more precisely, *forced one-dimensional (local) flows* of the form

$$\Xi: U \subseteq \mathbb{R} \times \mathbb{T}^D \times X \rightarrow \mathbb{T}^D \times X, \quad (t, \theta, x) \mapsto (t \cdot \rho + \theta, \xi(t, \theta, x)), \quad (1.1)$$

where $\rho \in \mathbb{R}^D$ and U is the domain of ξ which is the unique (under mild assumptions) maximal solution of

$$\partial_t \xi(t, \theta, x) = F(t \cdot \rho + \theta, \xi(t, \theta, x)) \quad (1.2)$$

with $\xi(0, \theta, x) = x$ for each $(\theta, x) \in \mathbb{T}^D \times X$. \mathbb{T}^D is called the *base space* or simply *base* of the flow Ξ in (1.1). We say the differential equation (1.2) as well as the flow Ξ are *driven by* ρ . Given ρ , we may further say Ξ is *generated by* F .

We throughout assume that ρ satisfies the following slow recurrence condition.

¹We should remark that in this linear setting, the existence of SNA's is equivalent to the *non-uniform hyperpolicity* of the respective cocycle, a property extensively studied in the literature. Results closely related to the ones of the present work can be found (though again, for discrete time systems) in [6, 20, 48].

Definition 1.1. We say $\rho \in \mathbb{R}^D$ is *Diophantine* (of type (\mathcal{C}, η)) if there are $\mathcal{C} > 0$ and $\eta \in \mathbb{R}$ such that

$$\forall k \in \mathbb{Z}^D \setminus \{0\}: \left| \sum_{i=1}^D \rho_i k_i \right| \geq \mathcal{C} |k|^{-\eta}.$$

Remark. Given $\eta_0 > D + 1$, it is well-known that almost every $\rho \in \mathbb{R}^D$ is Diophantine of type (\mathcal{C}, η_0) for some $\mathcal{C} > 0$.

We assume F to be C^2 in the following. In particular, this implies existence and uniqueness of $\xi(\cdot, \theta, x)$ for all $(\theta, x) \in \mathbb{T}^D \times X$ (on some non-degenerate time-interval containing 0) due to the Picard-Lindelöf Theorem (see, e.g. [19, Chapter II, Theorem 1.1 & 3.1]). Note that this yields

$$\xi(t + \tau, \theta, x) = \xi(t, \theta + \tau\rho, \xi(\tau, \theta, x)). \quad (1.3)$$

Reversing time, we introduce

$$\xi^- : (t, \theta, x) \mapsto \xi(-t, \theta, x) \quad (1.4)$$

which obviously solves (1.2) with the right-hand side replaced by

$$F^-(t \cdot \rho^- + \theta, \xi^-(t, \theta, x)), \quad (1.5)$$

where $\rho^- = -\rho$ and $F^- = -F$. Note that $\xi^-(t, t \cdot \rho + \theta, \xi(t, \theta, x)) = x$ because of (1.3).

Although it is standard, we want to mention that throughout this article, we slightly abuse notation by occasionally not distinguishing elements or subsets of \mathbb{T}^d from such of its cover \mathbb{R}^d ($d \in \mathbb{N}$). For example, we write $|\theta - \theta'|$ for the distance of $\theta, \theta' \in \mathbb{T}^d$ in the metric inherited from the Euclidean norm $|\cdot|$ in \mathbb{R}^d . Further, we identify the tangent space of \mathbb{T}^d at any θ with \mathbb{R}^d . When speaking of directional derivatives of $\xi(t, \cdot, x)$, we actually have in mind the respective derivatives of a lift of ξ (see [30, Definition A.1.19]). In this sense, given $\vartheta \in \mathbb{R}^d \setminus \{0\}$, it is clear what is to be understood by $\partial_{\vartheta} \xi(t, \theta, x)$. Higher derivatives are denoted and understood analogously. Typically, we consider directional derivatives with respect to a vector ϑ with $|\vartheta| = 1$ and write $\vartheta \in \mathbb{S}^{d-1}$ in this case.

1.2 Non-autonomous saddle-node bifurcations

An *invariant graph* of a local skew-product flow Ξ —we might occasionally speak of invariant graphs of F if F is the corresponding non-autonomous vector field and ρ is fixed—is a measurable function $\phi: \mathbb{T}^D \rightarrow X$ such that its graph $\Phi = \{(\theta, x): x = \phi(\theta)\}$ is invariant under Ξ , or equivalently,

$$\xi(t, \theta, \phi(\theta)) = \phi(t \cdot \rho + \theta) \quad (t \in \mathbb{R}).$$

By a slight abuse of notation, we refer by invariant graph to both the map ϕ as well as the corresponding point set $\Phi = \{(\theta, x) \in \mathbb{T}^D \times X: x = \phi(\theta)\}$ which we throughout denote by a capital letter. We identify invariant graphs which coincide $\text{Leb}_{\mathbb{T}^D}$ -almost everywhere, where $\text{Leb}_{\mathbb{T}^D}$ denotes Lebesgue measure on \mathbb{T}^D .

Associated to each invariant graph ϕ , there is an invariant ergodic measure μ_{ϕ} given by $\mu_{\phi}(A) = \text{Leb}_{\mathbb{T}^D}(\pi_1(A \cap \Phi))$ for each Borel set $A \subseteq \mathbb{T}^D \times X$, where $\pi_1: \mathbb{T}^D \times X \ni (\theta, x) \mapsto \theta$ denotes the canonical projection to the base coordinate. In fact, the converse is true as well: to each ergodic measure m there is an invariant graph ϕ such that $m = \mu_{\phi}$ (see [3, Theorem 1.8.4]). Hence, studying the invariant graphs of Ξ amounts to studying its ergodic measures.

Whether an invariant graph ϕ attracts or repels nearby orbits, is determined by its *Lyapunov exponent*

$$\lambda(\phi) = 1/t \cdot \int_{\mathbb{T}^D} \log |\partial_x \xi(t, \theta, \phi(\theta))| d\theta,$$

which is easily seen to be independent of the particular choice for $t > 0$. If $\lambda(\phi) < 0$, then ϕ is attracting and if $\lambda(\phi) > 0$, then ϕ is repelling (see [23, Proposition 3.3] for a precise statement).

Denote the set of non-autonomous C^2 -vector-fields on $\mathbb{T}^D \times X$ by $\mathcal{F}(X)$ (keeping the dimension D implicit). The set of C^2 -one-parameter families in $\mathcal{F}(X)$ is denoted by

$$\mathcal{P}(X) = \left\{ (F_\beta)_{\beta \in [0,1]} \mid F_\beta \in \mathcal{F}(X) \text{ for all } \beta \in [0,1] \text{ and } (\beta, \theta, x) \mapsto F_\beta(\theta, x) \text{ is } C^2 \right\}.$$

We may denote elements of $\mathcal{P}(X)$ also by $\hat{F} = (F_\beta)_{\beta \in [0,1]}$. We endow $\mathcal{P}(X)$ with the extended metric

$$d(\hat{F}, \hat{G}) = \sup_{\substack{(\theta, x) \in \mathbb{T}^D \times X \\ \beta \in [0,1]}} \sum_{\substack{s_1, s_2, s_3 \in \{0,1,2\} \\ s_1 + s_2 + s_3 \leq 2}} |\partial_\beta^{s_1} \partial_\theta^{s_2} \partial_x^{s_3} F_\beta(\theta, x) - \partial_\beta^{s_1} \partial_\theta^{s_2} \partial_x^{s_3} G_\beta(\theta, x)|.$$

With $\tilde{d} = d/(1+d)$, we may consider $\mathcal{P}(X)$ a metric space and refer to the respective topology as C^2 -topology in all of the following.

In this article, we study local bifurcations of invariant graphs, that is, given $\hat{F} \in \mathcal{P}(X)$ we investigate bifurcations of the corresponding graphs $(\phi_\beta)_{\beta \in [0,1]}$ which are assumed to be entirely contained² in a compact section $\Gamma = \mathbb{T}^D \times [\gamma^-, \gamma^+] \subseteq \mathbb{T}^D \times X$. In particular, we are interested in saddle-node bifurcations, that is, we study “collisions” of an attracting with a repelling graph. A natural setting for these collisions to occur is the subset $\mathcal{S}(X) \subseteq \mathcal{P}(X)$ where each $(F_\beta)_{\beta \in [0,1]} \in \mathcal{S}(X)$ satisfies the following assumptions for all $\beta \in [0,1]$ and $\theta \in \mathbb{T}^D$ if applicable

($\mathcal{S}1$) $F_\beta(\theta, \gamma^+) \leq 0$ and $F_\beta(\theta, \gamma^-) \leq 0$;

($\mathcal{S}2$) in Γ , there exist two invariant continuous graphs for F_0 but no invariant graph for F_1 ;

($\mathcal{S}3$) $\partial_\beta F_\beta(\theta, x) \leq 0$ and there is θ_0 such that $\partial_\beta F_\beta(\theta_0, x) < 0$ for all $x \in [\gamma^-, \gamma^+]$;

($\mathcal{S}4$) $\partial_x^2 F_\beta(\theta, x) < 0$ for all $x \in [\gamma^-, \gamma^+]$.

Here, ($\mathcal{S}1$)–($\mathcal{S}3$) guarantee that the two initial invariant graphs approach each other monotonously and collide (for a detailed discussion, see [2]). Assumption ($\mathcal{S}4$) ensures that there are at most two distinct invariant graphs and that the two invariant graphs of F_0 are attracting and repelling, respectively [2, Theorem 2.1]. Note that $\mathcal{S}(X)$ has non-empty interior in the C^2 -topology [41, Theorem 3.1].

The next statement describes the possible outcomes of saddle-node bifurcations in the present non-autonomous setting.

Theorem 1.2 (cf. [36, Theorem 3.1], [2, Theorem 7.1]). *Fix $\rho \in \mathbb{R}^D$ and $(F_\beta)_{\beta \in [0,1]} \in \mathcal{S}(X)$. There exists a unique critical parameter $\beta_c \in (0, 1)$ such that the following holds.*

- (i) *If $\beta < \beta_c$, then F_β has two invariant graphs $\phi_\beta^- < \phi_\beta^+$ in Γ , both of which are continuous. Further, $\lambda(\phi_\beta^-) > 0$ and $\lambda(\phi_\beta^+) < 0$.*

²We say an invariant graph ϕ is *entirely contained* in some set Γ if there is a representative $\tilde{\phi}$ in the equivalence class of ϕ whose graph satisfies $\tilde{\Phi} \subseteq \Gamma$.

(ii) If $\beta > \beta_c$, then F_β has no invariant graphs in Γ .

(iii) If $\beta = \beta_c$, then one of the following two alternatives holds.

Smooth bifurcation: F_{β_c} has a unique invariant graph ϕ_{β_c} in Γ , which satisfies $\lambda(\phi_{\beta_c}) = 0$. Either ϕ is continuous, or it contains both an upper and lower semi-continuous representative in its equivalence class.³

Non-smooth bifurcation: F_{β_c} has exactly two invariant graphs $\phi_{\beta_c}^- < \phi_{\beta_c}^+$ a.e. in Γ . The graph $\phi_{\beta_c}^-$ is lower semi-continuous, whereas $\phi_{\beta_c}^+$ is upper semi-continuous, but none of the graphs is continuous and there exists a residual set $\Omega \subseteq \mathbb{T}^d$ such that $\phi_{\beta_c}^-(\theta) = \phi_{\beta_c}^+(\theta)$ for all $\theta \in \Omega$.

The graphs appearing in a non-smooth bifurcation are the main theme of this article.

Definition 1.3. A non-continuous invariant graph ϕ is called a *strange non-chaotic attractor (SNA)* if $\lambda(\phi) < 0$; it is called a *strange non-chaotic repeller (SNR)* if $\lambda(\phi) > 0$.

1.3 Basic notions from fractal geometry

We will describe the geometry of the SNA's that arise in a non-smooth bifurcation in terms of some concepts from fractal geometry, which we introduce in this paragraph.

Let Y be a metric space. For $\varepsilon > 0$, we call a finite or countable collection $\{A_i\}$ of subsets of Y an ε -cover of A if $|A_i| \leq \varepsilon$ for each i and $A \subseteq \bigcup_i A_i$.

Definition 1.4. For $A \subseteq Y$, $s \geq 0$ and $\varepsilon > 0$, we define

$$\mathcal{H}_\varepsilon^s(A) = \inf \left\{ \sum_i |A_i|^s \mid \{A_i\} \text{ is an } \varepsilon\text{-cover of } A \right\}$$

and call

$$\mathcal{H}^s(A) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon^s(A)$$

the s -dimensional Hausdorff measure of A . The Hausdorff dimension of A is defined by

$$D_H(A) = \sup\{s \geq 0 \mid \mathcal{H}^s(A) = \infty\}.$$

Remark. Notice that D_H is obviously monotone, that is, if $A \subseteq B$, then $D_H(A) \leq D_H(B)$.

Definition 1.5. The lower and upper box-counting dimension of a totally bounded subset $A \subseteq Y$ are defined as

$$\underline{D}_B(A) = \liminf_{\varepsilon \rightarrow 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon},$$

$$\overline{D}_B(A) = \limsup_{\varepsilon \rightarrow 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon},$$

where $N(A, \varepsilon)$ is the smallest number of sets of diameter at most ε needed to cover A . If $\underline{D}_B(A) = \overline{D}_B(A)$, then we call their common value $D_B(A)$ the *box-counting dimension* (or *capacity*) of A .

³We call an invariant graph *continuous* if it allows for a continuous representative and similarly, we call it *semi-continuous* if it allows for a semi-continuous representative. Observe that hence, we call an invariant graph *non-continuous* if there is no continuous representative.

Lemma 1.6 ([12, Corollary 2.4]). *Let Y and Z be two metric spaces and assume that $g : A \subseteq Y \rightarrow Z$ is a bi-Lipschitz continuous map. Then $D_H(g(A)) = D_H(A)$.*

Theorem 1.7 ([22, Corollary 12] & [21, Corollary 4]). *Suppose Y and Z are two metric spaces and consider the Cartesian product space $Y \times Z$ equipped with the maximum metric. Then for $A \subseteq Y$ and $B \subseteq Z$ totally bounded, we have*

$$D_H(A \times B) \leq D_H(A) + \overline{D}_B(B) \quad \text{and} \quad D_H(A \times B) \geq D_H(A) + D_H(B).$$

Lemma 1.8 ([12]). *Given a totally bounded set with well-defined box-counting dimension $D_B(A)$, we have $D_B(A) = D_B(\overline{A})$. Moreover, if $A \subseteq \mathbb{R}^d$ and $\text{Leb}_{\mathbb{R}^d}(A) > 0$, then $D_B(A) = d$.*

Definition 1.9. For $d \in \mathbb{N}$, we call a Borel set $A \subseteq Y$ *countably d -rectifiable* if there exists a sequence of Lipschitz continuous functions $(g_i)_{i \in \mathbb{N}}$ with $g_i : A_i \subseteq \mathbb{R}^d \rightarrow Y$ such that $\mathcal{H}^d(A \setminus \bigcup_i g_i(A_i)) = 0$. A finite Borel measure μ is called *d -rectifiable* if $\mu = \Theta \mathcal{H}^d \llcorner_A$ for some countably d -rectifiable set A and some Borel measurable density $\Theta : A \rightarrow [0, \infty)$.

1.4 Existence and geometry of SNA's of forced interval maps

In this section, we review the situation for discrete time systems. Let us thus consider *qpf monotone interval maps*

$$f : \mathbb{T}^d \times X \rightarrow \mathbb{T}^d \times X, \quad (\theta, x) \mapsto (\theta + \omega, f_\theta(x)), \quad (1.6)$$

where $d \in \mathbb{N}$, X is as above, $f_\theta(\cdot)$ is monotonously increasing for each fixed θ , and $\omega \in \mathbb{T}^d$ satisfies the following slow-recurrence condition (compare to Definition 1.1).

Definition 1.10. We say $\omega \in \mathbb{T}^d$ is *Diophantine (of type (\mathcal{C}, η))* if there are $\mathcal{C} > 0$ and $\eta \in \mathbb{R}$ such that

$$\forall k \in \mathbb{Z}^d \setminus \{0\} : \sup_{p \in \mathbb{Z}} \left| \sum_{i=1}^d \omega_i k_i + p \right| \geq \mathcal{C} |k|^{-\eta}.$$

We call $f_\theta(\cdot)$ the *fibre map* corresponding to $\theta \in \mathbb{T}^d$. The notions of invariant graphs, the associated invariant measures, and SNA's are analogously defined as in the continuous-time case, with the Lyapunov exponent of an invariant graph $\tilde{\phi}$ given by

$$\int_{\mathbb{T}^d} \log \left| \partial_x f_\theta(\tilde{\phi}(\theta)) \right| d\theta.$$

Theorem 1.11 (cf. [14]). *Let $X \subseteq \mathbb{R}$ be a non-degenerate interval, suppose $\omega \in \mathbb{T}^d$ is Diophantine and consider the space of one-parameter families*

$$\mathcal{P}_\omega(X) = \left\{ (f_\beta)_{\beta \in [0,1]} : [0,1] \times \mathbb{T}^d \times X \ni (\beta, \theta, x) \mapsto (\theta + \rho, f_{\beta, \theta}(x)) \text{ is } C^2 \right\}$$

equipped with the C^2 -metric.⁴ There exists a non-empty open set $\mathcal{U}_\omega(X) \subseteq \mathcal{P}_\omega(X)$ such that each $(f_\beta)_{\beta \in [0,1]} \in \mathcal{U}_\omega(X)$ admits an SNA and an SNR for some $\beta_c \in (0, 1)$.

⁴Which is similarly defined as in the continuous time case.

We specify the set $\mathcal{U}_\omega(X)$ in Section 4 (see Theorem 4.4). It is essentially given by a number of C^2 -estimates on the map f restricted to a section $\mathbb{T}^d \times [\gamma^-, \gamma^+]$ (with $f_\beta(\gamma^\pm) \leq \gamma^\pm$) that contains the SNA/SNR-pair of the previous statement.

We next provide a finer geometric description of the SNA $\tilde{\phi}_{\beta_c}^+$ that occurs in the previous theorem and its corresponding ergodic measure $\mu_{\tilde{\phi}_{\beta_c}^+}$. We moreover give a simple description of the minimal set in the section $\Gamma = \mathbb{T}^d \times [\gamma^-, \gamma^+]$ as the *maximal invariant set (in Γ)*. For $\beta \in [0, 1]$, this is given by

$$\tilde{\Lambda}_\beta = \bigcap_{n \in \mathbb{Z}} f_\beta^n(\Gamma).$$

Note that $\tilde{\Lambda}_{\beta_c} \neq \emptyset$. It turns out that

$$\psi_{\tilde{\Lambda}_{\beta_c}}^-(\theta) = \inf \tilde{\Lambda}_{\beta_c}(\theta) \quad \text{and} \quad \psi_{\tilde{\Lambda}_{\beta_c}}^+(\theta) = \sup \tilde{\Lambda}_{\beta_c}(\theta)$$

are lower and upper semi-continuous representatives of $\tilde{\phi}_{\beta_c}^+$ and $\tilde{\phi}_{\beta_c}^-$.⁵

The next assertion is a crucial ingredient for our geometric analysis.

Proposition 1.12 (cf. [15]). *Let ω and $(f_\beta)_{\beta \in [0,1]}$ be as in Theorem 1.11. There is an increasing sequence of sets $\tilde{\Omega}_j \subseteq \mathbb{T}^d$ such that $\psi_{\tilde{\Lambda}_{\beta_c}}^+ \upharpoonright_{\tilde{\Omega}_j}$ is Lipschitz continuous for all $j \in \mathbb{N}$ and $D_H(\tilde{\Omega}_\infty) \leq d-1$, where $\tilde{\Omega}_\infty = \mathbb{T}^d \setminus \bigcup_{j \in \mathbb{N}} \tilde{\Omega}_j$. An analogous result holds for the repeller $\psi_{\tilde{\Lambda}_{\beta_c}}^-$.*

Theorem 1.13 (cf. [15]). *Let ω and $(f_\beta)_{\beta \in [0,1]}$ be as in Theorem 1.11. Then the SNA $\psi_{\tilde{\Lambda}_{\beta_c}}^+$ satisfies the following.*

- (i) $D_B(\Psi_{\tilde{\Lambda}_{\beta_c}}^+) = d+1$ and $D_H(\Psi_{\tilde{\Lambda}_{\beta_c}}^+) = d$.
- (ii) The measure $\mu_{\psi_{\tilde{\Lambda}_{\beta_c}}^+}$ is d -rectifiable.
- (iii) $\tilde{\Lambda}_{\beta_c}$ is minimal. We have $\tilde{\Lambda}_{\beta_c} = \overline{\Psi_{\tilde{\Lambda}_{\beta_c}}^-} = \overline{\Psi_{\tilde{\Lambda}_{\beta_c}}^+}$.
- (iv) $\psi_{\tilde{\Lambda}_{\beta_c}}^+$ is the only semi-continuous representative in its equivalence class.

Analogous results hold for the repeller $\psi_{\tilde{\Lambda}_{\beta_c}}^-$.

Remark. Part (ii) as well as the statement concerning the Hausdorff dimension are direct implications of Proposition 1.12 (cf. [15, Theorem 3.2]).

2 Main results

The primary work of this article is to show the following.

Theorem 2.1. *Let h be a non-increasing C^2 -bump-function $h: \mathbb{R}_{\geq 0} \rightarrow [0, 1]$ with $h'(0) = 0$, $h' \upharpoonright_{(0,R)} < 0$ (for some $R > 0$), $h''(0) < 0$, $h(y) = 0$ for all $y \geq R$, and $h(0) = 1$. Given Diophantine $\rho \in \mathbb{R}^D$ of type (\mathcal{C}, η) , there is $\mathcal{R} = \mathcal{R}(|\rho|, \mathcal{C}, \eta)$ such that the family of qpf skew product flows $(\Xi_\beta)_{\beta \in [0,1]}$ driven by ρ and generated by the non-autonomous vector fields*

$$F_\beta(\theta, x) = -bx^2 + b - \beta b/(1 - b^{-1/2}) \cdot h(|\theta|),$$

undergoes a non-smooth saddle-node bifurcation within $\mathbb{T}^D \times [-1, 1]$ if $R \geq \mathcal{R}$ and b is sufficiently large.

⁵The semi-continuity is a general fact (see [42]), the other statement follows from the definition of $\mathcal{U}_\omega(X)$ (see [13])

In fact, we prove more: we show that for large enough b , the family $(\Xi_\beta)_{\beta \in [0,1]}$ can be reduced to a family of qpf monotone interval maps which lies in the set $\mathcal{U}_\omega(X)$ of Theorem 1.11 (for appropriate ω and X [see Section 3.2]). This yields the following corollary.

Theorem 2.2. *Suppose $\rho \in \mathbb{R}^D$ is Diophantine. Then there is a set $\mathcal{U}_\rho(X) \subseteq \mathcal{P}(X)$ with non-empty interior in the C^2 -topology such that each family of flows $(\Xi_\beta)_{\beta \in [0,1]}$ driven by ρ and generated by some $(F_\beta)_{\beta \in [0,1]} \in \mathcal{U}_\rho(X)$ undergoes a non-smooth saddle-node bifurcation.*

Coming back to the particular vector fields of Theorem 2.1, observe that we immediately get an analogous result for the skew product flow family generated by

$$L_\beta(\theta, x) = 2/r \cdot bx \cdot (r - x) - \beta b/(1 - b^{-1/2}) \cdot h(|\theta|),$$

for some $r > 0$. In other words, Theorem 2.1 indeed guarantees the occurrence of a non-smooth saddle-node bifurcation for the logistic differential equation with a (certain) quasiperiodic harvesting term.

In [7], a similar Riccati equation as in Theorem 2.1 is considered. There as well, the existence of SNA's is proven, however, in a regime already beyond the saddle-node bifurcation. Hence, in the words of the situation considered in [7], the novelty of Theorem 2.1 is that we describe the *transition* from uniform to non-uniform hyperbolicity. On a technical level, this difference is most visibly reflected in the fact that in the present work we have to cope with second derivatives of the flow.

Note that by the application of Theorem 1.11, our approach focusses on the geometry of the mechanism by which SNA's are created. This geometric insight shows that even if in general, analytical results on the occurrence of non-smooth bifurcations for particular ode's might still be subject to rather technical considerations, the proof in Section 4 should basically be extendable to situations with non-autonomous vector fields similar to the one above (for example, we believe that it should be possible to replace h by an arbitrary C^2 -function [on \mathbb{T}^D] with a unique non-degenerate global maximum). In a nutshell, it is not so much the particular choice of the vector fields F_β but the assumption of general features—like the concavity of the functions $F_\beta(\theta, \cdot)$ and the decreasing dependence on β —which guarantee a non-smooth saddle-node bifurcation (see Figure 1).

Finally, another merit of the reduction to the discrete time setting is that we can—with only a little extra work—carry over Theorem 1.13 to the continuous time setting. Hence, we obtain a fairly comprehensive description of the geometry of the SNA and the maximal invariant set in $\Gamma = \mathbb{T}^D \times [\gamma^-, \gamma^+]$ (recall that we are dealing with local bifurcations). For $\beta \in [0, 1]$, this is—similarly to the discrete time case—given by

$$\Lambda_\beta = \bigcap_{t \in \mathbb{R}} \Xi_\beta(t, \Gamma).$$

Note that Λ_β is non-empty for $\beta \leq \beta_c$ due to Theorem 1.2. Analogously to the discrete time case, we have that

$$\phi_{\Lambda_\beta}^-(\theta) = \inf \Lambda_\beta(\theta) \quad \text{and} \quad \phi_{\Lambda_\beta}^+(\theta) = \sup \Lambda_\beta(\theta)$$

are lower and upper semi-continuous invariant graphs, respectively, and hence representatives of the invariant graphs that appear along the saddle-node bifurcation of $(\Xi_\beta)_{\beta \in [0,1]}$.

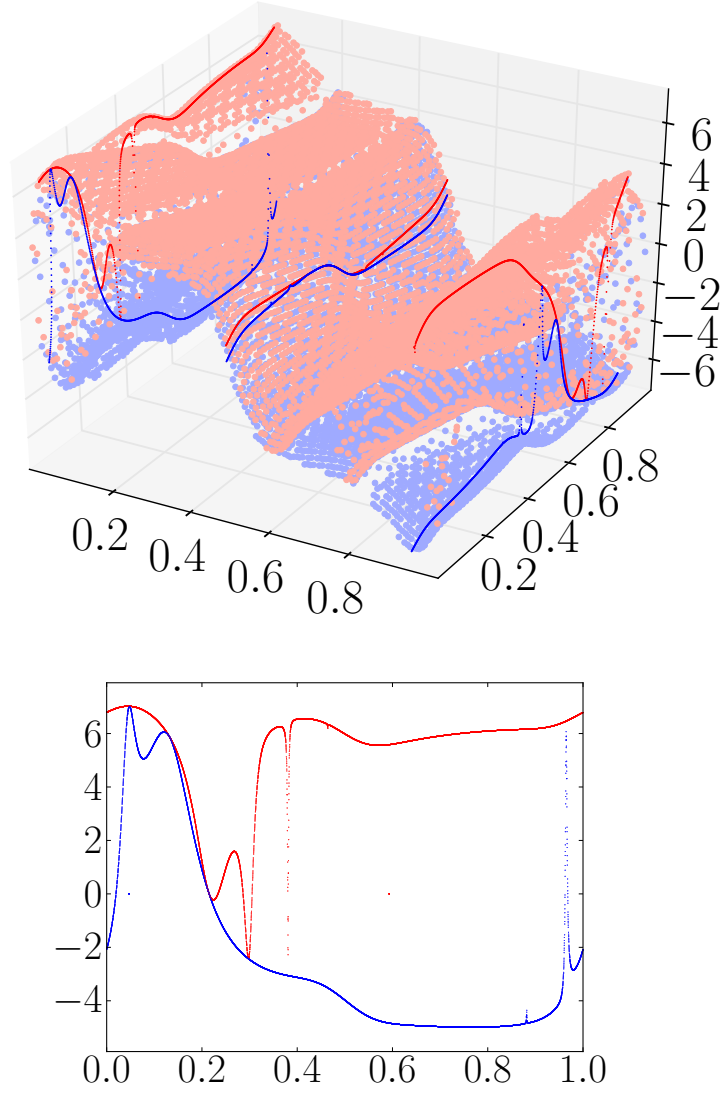


Figure 1: Invariant graphs of a skew product flow with $D = 2$ close to a non-smooth saddle-node bifurcation. The considered family of vector fields is given by $F_\beta(\theta, x) = -x^2 + b - \beta \cdot (2 - \cos^{11}(2\pi\theta_1) - \cos^{11}(2\pi\theta_2))/4$ where $\theta = (\theta_1, \theta_2)$. We put $\rho = ((\sqrt{5} - 1)/2, \pi)$, $b = 100$, and $\beta = 176.01538$. In the upper picture, the attractor is pale red, the repeller is depicted in pale blue. The curves in deep red and blue show slices of the attractor and repeller, respectively, for three different fixed values of θ_1 . The lower picture allows a closer look at the section close to $\theta_1 = 0$.

Theorem 2.3. Let ρ and $(\Xi_\beta)_{\beta \in [0,1]}$ be as in Theorem 2.2. Then the SNA $\phi_{\Lambda_{\beta_c}}^+$ appearing at the critical parameter β_c satisfies the following.

- (i) $D_B(\Phi_{\Lambda_{\beta_c}}^+) = D + 1$ and $D_H(\Phi_{\Lambda_{\beta_c}}^+) = D$.
- (ii) The measure $\mu_{\phi_{\Lambda_{\beta_c}}^+}$ is D -rectifiable.
- (iii) Λ_{β_c} is minimal. We have $\Lambda_{\beta_c} = \overline{\Phi_{\Lambda_{\beta_c}}^-} = \overline{\Phi_{\Lambda_{\beta_c}}^+}$.
- (iv) $\phi_{\Lambda_{\beta_c}}^+$ is the only semi-continuous representative in its equivalence class.

Analogous results hold for the repeller $\phi_{\Lambda_{\beta_c}}^-$.

Remark. Note that D -rectifiability of a measure μ implies that μ is exact dimensional with the point-wise dimension equal to D [1]. As a result of this, several other dimensions of μ coincide with D [49]. In particular, this is true for the information dimension [47].

3 Prerequisites

As already pointed out, the overall strategy of this article is to reduce particular skew product flows to qpf monotone interval maps in order to extend both Theorem 1.11 and Theorem 1.13 to qpf ode's. Appropriate Poincaré sections (and their corresponding return maps) which are suitable for this reduction are introduced in the second paragraph of this section.

As the application of the discrete time results involves to deal with a number of C^2 -estimates, we carry out some straightforward computations to provide the derivatives of $(\beta, t, \theta, x) \mapsto \xi_\beta(t, \theta, x)$ in the next paragraph.

3.1 Derivatives of the flow

It is well known (see, e.g. [19, Chapter V, Corollary 4.1]) that for $\hat{F} \in \mathcal{P}(X)$, the map $(\beta, t, \theta, x) \mapsto \xi_\beta(t, \theta, x)$ is C^2 so that the task here is to differentiate (1.2) and express the solutions of the resulting ode's (sometimes referred to as *variational equations*) in terms of the (unknown) solution ξ_β of (1.2). By differentiating (1.2) with respect to x and θ , we get

$$\partial_t \partial_x \xi_\beta(t, \theta, x) = \partial_x F_\beta(t\rho + \theta, \xi_\beta(t, \theta, x)) \cdot \partial_x \xi_\beta(t, \theta, x), \quad (3.1)$$

$$\partial_t \partial_\theta \xi_\beta(t, \theta, x) = \partial_\theta F_\beta(t\rho + \theta, \xi_\beta(t, \theta, x)) + \partial_x F_\beta(t\rho + \theta, \xi_\beta(t, \theta, x)) \cdot \partial_\theta \xi_\beta(t, \theta, x). \quad (3.2)$$

Further, note that since $\xi_\beta(0, \theta, x) = x$, we have $\partial_x \xi_\beta(0, \theta, x) = 1$ and $\partial_\theta \xi_\beta(0, \theta, x) = 0$. Hence,

$$\partial_x \xi_\beta(t, \theta, x) = \exp\left(\int_0^t \partial_x F_\beta(s\rho + \theta, \xi_\beta(s, \theta, x)) ds\right) \quad (3.3)$$

and

$$\partial_\theta \xi_\beta(t, \theta, x) = \int_0^t \partial_\theta F_\beta(s\rho + \theta, \xi_\beta(s, \theta, x)) \exp\left(\int_s^t \partial_x F_\beta(\tau\rho + \theta, \xi_\beta(\tau, \theta, x)) d\tau\right) ds. \quad (3.4)$$

The expression for $\partial_x \xi_\beta(t, \theta, x)$ immediately shows monotonicity of ξ_β in x . However, observe that this already follows from the uniqueness of the solutions of (1.2), of course. We can differentiate (3.3) to get

$$\partial_x^2 \xi_\beta(t, \theta, x) = \partial_x \xi_\beta(t, \theta, x) \cdot \int_0^t \partial_x^2 F_\beta(s\rho + \theta, \xi_\beta(s, \theta, x)) \cdot \partial_x \xi_\beta(s, \theta, x) ds \quad (3.5)$$

and similarly

$$\begin{aligned} \partial_\vartheta \partial_x \xi_\beta(t, \theta, x) \\ = \partial_x \xi_\beta(t, \theta, x) \cdot \int_0^t \partial_\vartheta \partial_x F_\beta(s\rho + \theta, \xi_\beta(s, \theta, x)) + \partial_x^2 F_\beta(s\rho + \theta, \xi_\beta(s, \theta, x)) \cdot \partial_\vartheta \xi_\beta(s, \theta, x) ds. \end{aligned} \quad (3.6)$$

For simplicity, instead of further differentiating (3.4) with respect to ϑ , we differentiate (3.2) and solve the resulting problem with initial condition $\partial_\vartheta^2 \xi_\beta(0, \theta, x) = 0$ in order to obtain an expression for $\partial_\vartheta^2 \xi_\beta(t, \theta, x)$. Now,

$$\begin{aligned} \partial_t \partial_\vartheta^2 \xi_\beta(t, \theta, x) &= \partial_\vartheta^2 F_\beta(t\rho + \theta, \xi_\beta(t, \theta, x)) + 2\partial_\vartheta \partial_x F_\beta(t\rho + \theta, \xi_\beta(t, \theta, x)) \cdot \partial_\vartheta \xi_\beta(t, \theta, x) \\ &\quad + \partial_x^2 F_\beta(t\rho + \theta, \xi_\beta(t, \theta, x)) \cdot (\partial_\vartheta \xi_\beta(t, \theta, x))^2 \\ &\quad + \partial_x F_\beta(t\rho + \theta, \xi_\beta(t, \theta, x)) \cdot \partial_\vartheta^2 \xi_\beta(t, \theta, x). \end{aligned}$$

The solution is straightforwardly given by

$$\begin{aligned} \partial_\vartheta^2 \xi_\beta(t, \theta, x) &= \\ &\int_0^t \left[\partial_x^2 F_\beta(s\rho + \theta, \xi_\beta(s, \theta, x)) (\partial_\vartheta \xi_\beta(s, \theta, x))^2 + \partial_\vartheta^2 F_\beta(s\rho + \theta, \xi_\beta(s, \theta, x)) \right. \\ &\quad \left. + 2\partial_x \partial_\vartheta F_\beta(s\rho + \theta, \xi_\beta(s, \theta, x)) \partial_\vartheta \xi_\beta(s, \theta, x) \right] \exp\left(\int_s^t \partial_x F_\beta(\tau\rho + \theta, \xi_\beta(\tau, \theta, x)) d\tau\right) ds. \end{aligned} \quad (3.7)$$

3.2 Reduction to a Poincaré map

Let us drop the index β in this paragraph and set $d = D - 1$.

Assume without loss of generality that $|\rho_D| = \max_{j=1, \dots, D} |\rho_j|$. Note that since $\rho = (\rho_1, \dots, \rho_D) \in \mathbb{R}^D$ is Diophantine of type (\mathcal{C}, η) (see Definition 1.1), we have that $\omega = \omega(\rho) = (\rho_1/\rho_D, \dots, \rho_d/\rho_D) \in \mathbb{T}^d$ is Diophantine of type (\mathcal{C}', η') (see Definition 1.10) with $\eta' = \eta$ and \mathcal{C}' proportional to \mathcal{C}/ρ_D .

Before we can reduce the *local* flow Ξ from (1.1) to a skew product of the form (1.6), we need to make it a flow, that is, we need the set U to equal $\mathbb{R} \times \mathbb{T}^D \times X$ so that any point in $\mathbb{T}^D \times X$ has a full orbit. To that end, recall that we are dealing with local bifurcations occurring in a section $\Gamma = \mathbb{T}^D \times [\gamma^-, \gamma^+] \subseteq \mathbb{T}^D \times X$ and assume, for simplicity, that $[\gamma^-, \gamma^+]$ is in the interior of X . By changing the non-autonomous vector field F outside of Γ , we obviously do not change anything about the considered bifurcation within Γ . Hence, we may replace F by $\tilde{F} = h \cdot F$, where $h: \Theta \times X \ni (\theta, x) \mapsto \tilde{h}(x) \in [0, 1]$ is a smooth function with $\tilde{h}|_{[\gamma^-, \gamma^+]} = 1$ and $\tilde{h}|_{X \setminus [\gamma^- - 2\varepsilon, \gamma^+ + 2\varepsilon]} = 0$ for some $\varepsilon > 0$ with $[\gamma^- - 3\varepsilon, \gamma^+ + 3\varepsilon] \subseteq X$. With \tilde{F} , we actually have a flow since a given orbit either stays within $[\gamma^- - 2\varepsilon, \gamma^+ + 2\varepsilon]$ or is eventually constant so that every orbit is well-defined for all times. In the following, we will not stress this detail. However, the reader should always think of the modified vector field \tilde{F} whenever full orbits are assumed for arbitrary initial conditions.

In this sense, consider the first return map to the Poincaré section $\mathbb{T}_D^d = \{\theta \in \mathbb{T}^D : \theta_D = 0\}$, that is, the map

$$\begin{aligned} \tilde{\Xi}: \mathbb{T}_D^d \times X &\rightarrow \mathbb{T}_D^d \times X, \\ (\theta, x) &\mapsto \Xi(1/\rho_D, \theta, x) = \left(\theta + 1/\rho_D \cdot \rho, \tilde{\xi}_\theta(x)\right), \end{aligned} \quad (3.8)$$

where $\tilde{\xi}_\theta(x) = \xi(1/\rho_D, \theta, x)$. Note that (3.8) is of the form (1.6). From now on, we identify \mathbb{T}_D^d with \mathbb{T}^d and thus consider \mathbb{T}^d a subset of \mathbb{T}^D (slightly abusing terminology). In a similar fashion, we may write $\theta + \omega$ when $\theta \in \mathbb{T}^D$ and actually $\theta + 1/\rho_D \cdot \rho$ is meant.

It is obvious that an invariant graph of the flow Ξ yields an invariant graph for its first return map $\tilde{\Xi}$. The following basic observation provides us with a converse.

Proposition 3.1. *Consider the flow Ξ in (1.1) with a non-autonomous C^1 -vector field F and suppose there is an invariant graph $\tilde{\phi}: \mathbb{T}^d \rightarrow X$ for the corresponding first return map $\tilde{\Xi}$. Then there is a unique invariant graph ϕ for Ξ with $\phi(\theta) = \tilde{\phi}(\theta)$ for each $\theta \in \mathbb{T}^d$ and ϕ is continuous if and only if $\tilde{\phi}$ is continuous. Further, if $\tilde{\Phi}$ is relatively compact in $\mathbb{T}^d \times X$, then $\lambda(\phi) = \rho_D \cdot \lambda(\tilde{\phi})$.*

Proof. For $\theta \in \mathbb{T}^D$, set $t_\theta = \theta_D / \rho_D$. Then, $\phi: \theta \mapsto \xi(t_\theta, \theta - t_\theta \rho, \tilde{\phi}(\theta - t_\theta \rho))$ is invariant under Ξ . The uniqueness and the assertion about the continuity are obvious.

Now, note that if $\tilde{\Phi}$ is compact in X , then so is $\bar{\Phi} \subseteq \Xi([0, 1] \times \tilde{\Phi})$. As F is C^1 , $\partial_x F$ is thus bounded on $\bar{\Phi}$ and hence integrable. By means of (3.3), we therefore have

$$\begin{aligned} \lambda(\tilde{\phi}) &= \int_{\mathbb{T}^d} \log |\partial_x \tilde{\xi}_\theta(\tilde{\phi}(\theta))| d\theta = \int_{\mathbb{T}^d} \int_0^{1/\rho_D} \partial_x F_\beta(\theta + s\rho, \xi_\beta(s, \theta, \tilde{\phi}(\theta))) ds d\theta \\ &= \int_{\mathbb{T}^d} \int_0^{1/\rho_D} \partial_x F_\beta(\theta + s\rho, \phi(\theta + s\rho)) ds d\theta = 1/\rho_D \cdot \int_{\mathbb{T}^D} \partial_x F_\beta(\theta, \phi(\theta)) d\theta \end{aligned}$$

and hence

$$\begin{aligned} \lambda(\phi) &= \rho_D \cdot \int_{\mathbb{T}^D} \log |\partial_x \xi(1/\rho_D, \theta, \phi(\theta))| d\theta \\ &= \rho_D \cdot \int_{\mathbb{T}^1} \int_{\mathbb{T}^d} \int_0^{1/\rho_D} \partial_x F_\beta(\theta + s\rho, \phi(\theta + s\rho)) ds d(\theta_1, \dots, \theta_d) d\theta_D \\ &= \int_{\mathbb{T}^1} \int_{\mathbb{T}^D} \partial_x F_\beta(\theta, \phi(\theta)) d\theta d\theta_D = \rho_D \cdot \lambda(\tilde{\phi}). \end{aligned} \quad \square$$

4 The quasiperiodically driven logistic differential equation

Let us consider a one-parameter family of skew product flows Ξ_β of the form (1.1) with $X = \mathbb{R}$ and

$$F_\beta(\theta, x) = -bx^2 + b - \beta b / (1 - b^{-1/2}) \cdot g(\theta), \quad (*)$$

where $b > 1$ and $g: \mathbb{T}^D \rightarrow [0, 1]$ is C^2 and assumes a unique non-degenerate global maximum at some $\bar{\theta} \in \mathbb{T}^D$. Without loss of generality, we may assume that $g(\bar{\theta}) = 1$. To reduce the technicalities of our investigation to a minimum, we assume further that $g(\theta) = h(|\theta - \bar{\theta}|)$, where h is a non-increasing C^2 -bump-function $h: \mathbb{R}_{\geq 0} \rightarrow [0, 1]$ with $h'(0) = 0$, $h' \upharpoonright_{(0, R)} < 0$ (for some $R > 0$), $h''(0) < 0$, $h(y) = 0$ for all $y \geq R$, and $h(0) = 1$.

It is not hard to see that $(F_\beta)_{\beta \in [0, 1]}$ lies in $\mathcal{P}(\mathbb{R})$ and satisfies $(\mathcal{S}1)$ – $(\mathcal{S}4)$ with $\gamma^+ = 1$ and $\gamma^- = -1$ (where $(\mathcal{S}2)$ follows from Claim 4.1.1 below), that is, $(*)$ undergoes a saddle-node bifurcation in the sense of Theorem 1.2. In fact, this is true for any section containing $\mathbb{T}^D \times [\gamma^-, \gamma^+]$. Our goal is to show that there is \mathcal{R} (independent of b) such that if $R \geq \mathcal{R}$ and b is sufficiently large, then $(*)$ undergoes a *non-smooth* bifurcation.

By the previous section, we hence have to show that the first return maps $(\tilde{\xi}_\beta)_{\beta \in [0, 1]}$ corresponding to $(*)$ lie in the set $\mathcal{U}_\omega(X)$ of Theorem 1.11. Recall that we have not explicitly defined the set $\mathcal{U}_\omega(X)$ so far. It is essentially given by a list of estimates, denoted by $(\mathcal{A}1)$ – $(\mathcal{A}15)$, on the fibre maps and their first as well as second derivatives. In the following, we will provide these estimates already adapted to the first return maps $(\tilde{\Xi}_\beta)_{\beta \in [0, 1]}$ and concurrently prove that they are actually verified in

the present situation. Note that there are subtle differences to the presentation in [14]. The reader interested in the details may consult [13].

Let us introduce some notation. Given θ and θ' in \mathbb{T}^D such that there is $s \in [0, 1/\rho_D]$ with $\theta' = \theta + s\rho$, we denote by $[\theta, \theta']$ the line segment $\{\theta + \tau\rho : \tau \in [0, s]\}$. Similarly, given $A, B \subseteq \mathbb{T}^D$ such that for all $\theta \in A$ there exists a unique $s(\theta) \in [0, 1/\rho_D]$ so that $B = \{\theta + s(\theta)\rho : \theta \in A\}$, we set $[A, B] = \bigcup_{\theta \in A} [\theta, \theta + s(\theta)\rho]$.

We suppose there is $\delta_1 > 0$ much smaller than $1/\rho_D$ (in fact, $\delta_1 < \min\{1/18, 1/(36\rho_D)\}$ is sufficient) such that $[\mathbb{T}^d, \mathbb{T}^d - \delta_1\rho] \cap B_R(\bar{\theta}) = \emptyset$ where $B_R(\bar{\theta})$ denotes the ball of radius R around $\bar{\theta}$, that is, in one iteration, the time span before an orbit hits the bump is much bigger than the time after hitting the bump. We may further assume there is a positive constant $\delta_2 < \delta_1$ such that $[\mathbb{T}^d - \delta_2\rho, \mathbb{T}^d] \cap B_R(\bar{\theta}) = \emptyset$. By possibly shifting the $\theta_D = 0$ section, both assumptions boil down to assuming that R is small (independently of b).

To establish the existence of an attracting and a repelling invariant graph, we need regions where the dynamics are contracting and expanding, respectively. We will locate these regions in an interval of contraction $C = [1 - c, 1 + c]$ (for some positive $c < 1/4$) and an interval of expansion $E = [-1, -1 + \exp(-b/(2\rho_D))]$.⁶ In the following, we restrict our analysis to the section $\Gamma = \mathbb{T}^d \times [-1, 1 + c]$.

Although in principle, we could consider the flows corresponding to $(*)$ for all $\beta \in [0, 1]$, we will show below that if β is too close to 1, there are θ and x such that there exist $t_- < t_+ \in \mathbb{R}$ with $\lim_{t \rightarrow t_\pm} \Xi_\beta(t, \theta, x) = \mp\infty$. Such solutions clearly rule out the existence of an invariant graph in Γ . For that reason, setting

$$\begin{aligned} \beta_- &= \min\{\beta \in [0, 1] : \exists \theta \in \mathbb{T}^d \text{ such that } \tilde{\xi}_{\beta, \theta}(1 - c) \leq -1 + \exp(-b/(2\rho_D))\}, \\ \beta_+ &= \max\{\beta \in [0, 1] : \tilde{\xi}_{\beta, \theta}(1 + c) \geq -1 \text{ for all } \theta \in \mathbb{T}^d\}, \end{aligned}$$

all of the following assumptions are only supposed to hold for all $\beta \in [0, \beta_+]$ (if applicable).

Finally, we define the *critical region*

$$\mathcal{I}_0 = \left\{ \theta \in \mathbb{T}^d : [\theta, \theta + \omega] \cap \overline{B_R(\bar{\theta})} \neq \emptyset \right\}$$

and introduce the following constants which will serve as bounds on our derivatives

$$\alpha_c^{-1} = \alpha_e = \exp[2b(1 - c)(1/\rho_D - \delta_1) - 10b\delta_1] \quad \text{and} \quad \alpha_l^{-1} = \alpha_u = \exp[2b(1 + c)/\rho_D].$$

With these definitions, we are now in a position to go through the assumptions that define the set $\mathcal{U}_\omega(X)$. Let us first consider $(\mathcal{A}1)$ – $(\mathcal{A}8)$.

$$(\mathcal{A}1) \quad 0 < \partial_x \tilde{\xi}_{\beta, \theta}(x) < \alpha_c \text{ for } (\theta, x) \in \mathbb{T}^d \times C;$$

$$(\mathcal{A}2) \quad \partial_x \tilde{\xi}_{\beta, \theta}(x) > \alpha_e \text{ for } (\theta, x) \in (\mathbb{T}^d \setminus \mathcal{I}_0) \times E \cap \tilde{\Xi}_\beta^{-1}(\mathbb{T}^d \times E);$$

$$(\mathcal{A}3) \quad \alpha_l < \partial_x \tilde{\xi}_{\beta, \theta}(x) < \alpha_u \text{ for all } (\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma).$$

Observe that the above assumptions justify the naming of the intervals C and E .

The mechanism by which the SNA/SNR-pair is created in Theorem 1.11 is essentially the following: First, the existence of a continuous attractor and a continuous repeller is guaranteed for small β . Then, by increasing β , we move these two initial invariant graphs closer and closer to each other on a small set until they finally touch on a $\text{Leb}_{\mathbb{T}^d}$ -null set. This yields the desired discontinuity. The existence of the initial invariant graphs is ensured by

⁶Observe that by choosing C to lie above E , we decided the repeller to be below the attractor.

$$(\mathcal{A4}) \quad \tilde{\xi}_{\beta,\theta}(1+c) \leq 1+c \text{ and } \tilde{\xi}_{\beta,\theta}(-1) \leq -1$$

and the first part of $(\mathcal{A8})$ below. In order to make the invariant graphs touch each other, we have to connect the regions of contraction and expansion more and more with growing β which is why we assume a certain monotonicity in the dependence of β (see $(\mathcal{A9})$). The connection between C and E , however, needs to be realised carefully in order to guarantee that the two graphs only touch on a measure zero set. To that end, we only allow orbits starting within the critical region to pass from C to E

$$(\mathcal{A5}) \quad \tilde{\xi}_{\beta,\theta}(x) \in C \text{ for all } x \in [-1 + \exp(-b/(2\rho_D)), 1+c] \text{ and } \theta \notin I_0.$$

The set which contains (at least) all θ for which $\tilde{\xi}_{\beta,\theta}(1-c) \leq -1 + \exp(-b/(2\rho_D))$ is denoted by $\mathcal{J}_{0,\beta}$ and is obviously a subset of I_0 . We want it to satisfy

$$(\mathcal{A6}) \quad \mathcal{J}_{0,\beta} \text{ is closed and convex and } \mathcal{J}_{0,\beta} \subseteq \mathcal{J}_{0,\beta'} \text{ for each } \beta' \geq \beta;$$

$$(\mathcal{A7}) \quad \partial_{\theta}^2 \tilde{\xi}_{\beta,\theta}(x) > s \text{ for each } \theta \in \mathbb{S}^{d-1}, x \in C \text{ and all } \theta \in \mathcal{J}_{0,\beta}.$$

$$(\mathcal{A8}) \quad \tilde{\xi}_{\beta,\theta}(1-c) \geq 1-c \text{ for all } \theta \in \mathbb{T}^d \text{ and } \tilde{\xi}_{\beta_+,\theta}(1+c) \leq -1 \text{ for some } \theta \in \mathbb{T}^d;$$

$$(\mathcal{A9}) \quad \tilde{\xi}_{(\cdot)}(\theta, x) \text{ is non-increasing for fixed } (\theta, x) \in \Gamma.$$

Before we come to prove $(\mathcal{A1})$ – $(\mathcal{A9})$, we provide some simple observations. From now on, given ρ , we denote by $\theta_0 \in \mathbb{T}^d$ that point which passes through the maximum of g in $\bar{\theta}$ within one time step, that is, $\bar{\theta} \in [\theta_0, \theta_0 + \omega]$.

Proposition 4.1. *Suppose F_β is given by $(*)$. Then*

$$(a) \quad \xi_\beta(t, \theta, x) < x \text{ if } |x| > 1 \text{ where } t > 0, \theta \in \mathbb{T}^D, \text{ and } \beta \in [0, 1].$$

$$(b) \quad \xi_\beta(t, \theta, x) \geq x \text{ if } |x| \leq 1 \text{ where } \theta \in \mathbb{T}^D, t \in [0, 1/\rho_D] \text{ is such that } [\theta, \theta + t\rho] \cap B_R(\bar{\theta}) = \emptyset, \text{ and } \beta \in [0, 1].$$

Proof. (a) follows easily from the fact that $F_\beta(\theta, x) < 0$ for arbitrary β, θ , and $|x| > 1$. Likewise, (b) follows from the fact that $F_\beta(\theta, x) \geq 0$ for arbitrary $\beta, \theta \notin B_R(\bar{\theta})$, and $|x| \leq 1$. \square

Now, let us consider $(\mathcal{A1})$ – $(\mathcal{A9})$ in opposite order. $(\mathcal{A9})$ follows immediately from the monotone dependence of $(*)$ on β . The first part of $(\mathcal{A8})$ is immediate. The existence of $\beta_+ \in (0, 1)$ such that the second estimate of $(\mathcal{A8})$ holds true follows from the next statement under the assumption of sufficiently large b .

It is convenient to introduce $U_\varepsilon = \{\theta \in \mathbb{T}^D : g(\theta) \geq 1 - \varepsilon\}$ where $\varepsilon > 0$. Clearly, U_ε is nothing but $B_{h^{-1}([1-\varepsilon, 1])}(\bar{\theta})$.

Claim 4.1.1. *Suppose F_β is given by $(*)$ and b is sufficiently large. Then there exists $\beta \in (0, 1)$ such that $\xi_\beta(t, \theta_0, 1+c)$ is well-defined for all $t \in [0, 1/\rho_D]$ and $\tilde{\xi}_{\beta,\theta_0}(1+c) \leq -1$.*

Proof. Note that for t with $\theta_0 + t\rho \in U_{b^{-1/2}/2}$, we have $\partial_t \xi_\beta(t, \theta_0, 1+c) = -b\xi_\beta(t, \theta_0, 1+c)^2 + b - \beta b/(1 - b^{-1/2})g(\theta_0 + t\rho) \leq -b\xi_\beta(t, \theta_0, 1+c)^2 - b(\beta/(2b^{1/2} - 2) + \beta - 1)$. Consider such β for which $\beta/(2b^{1/2} - 2) + \beta - 1 > 0$. Observe that

$$y_\beta(t) = -\sqrt{\beta/(2b^{1/2} - 2) + \beta - 1} \tan \left(b \sqrt{\beta/(2b^{1/2} - 2) + \beta - 1} (t - t_0) + \alpha \right)$$

with $\alpha = \arctan\left(-(1+c)/\sqrt{\beta/(2b^{1/2}-2)+\beta-1}\right)$ solves

$$\partial_t y_\beta(t) = -by_\beta(t)^2 - b(\beta/(2b^{1/2}-2) + \beta - 1)$$

with $y_\beta(t_0) = 1+c$. Thus, $y_\beta(t)$ is an upper bound for $\xi_\beta(t, \theta_0, 1+c)$ for all $t \in [t_0, t_1]$, where $[t_0, t_1]$ is set to be the maximal interval with $[\theta_0 + t_0\rho, \theta_0 + t_1\rho] \subseteq U_{b^{-1/2}/2}$. Note that $|t_1 - t_0| > b^{-1/2}$ for big enough b since g assumes a maximum in $\bar{\theta}$. Now, for large enough b , there is $\beta \in (0, 1)$ such that $y_\beta(b^{-1/2} + t_0) < -1$ which proves that the image of $[0, 1] \ni \beta \mapsto \xi_\beta(t_1, \theta_0, 1+c)$ contains $[-1, 1]$. Proposition 4.1(a) and the continuous dependence of $\tilde{\xi}_{\beta, \theta_0}(1+c)$ on β hence yield the statement. \square

(A6) and (A7) are treated in Lemma 4.3. For sufficiently large b , (A5) is a consequence of the following statement.

Claim 4.1.2. *Suppose F_β is given by (*), $\theta \notin \mathcal{I}_0$ and b is sufficiently large. Then $\tilde{\xi}_{\beta, \theta}(-1 + \exp[-b/(2\rho_D)]) > 1 - c$.*

Proof. Note that as $\theta \notin \mathcal{I}_0$, we have that $\xi_\beta(t, \theta, -1 + \exp(-b/(2\rho_D)))$ equals $y(t)$ for $t \in [0, 1/\rho_D]$, where y is the solution of the initial value problem

$$\dot{y} = -by^2 + b, \quad y(0) = -1 + \exp(-b/(2\rho_D)).$$

Now, $y(t) = \tanh(b \cdot t + \alpha)$, where $|\alpha| = |\operatorname{artanh}(-1 + \exp[-b/(2\rho_D)])| \leq b/(3\rho_D)$. Hence, $y(1/\rho_D) \geq \tanh(2b/(3\rho_D)) > 1 - c$ for large enough b . \square

(A4) is an immediate consequence of Proposition 4.1 (a). Hence, apart from (A6) and (A7), we are left with (A1)–(A3) which follow from the next assertion.

Proposition 4.2. *Suppose F_β is given by (*) and b is sufficiently large. Then*

- (a) $\partial_x \tilde{\xi}_{\beta, \theta}(x) \leq \exp(-2b(1-c)(1/\rho_D - \delta_1) + 4b\delta_1)$ and $\tilde{\xi}_{\beta, \theta}(x) > -2$ for $(\theta, x) \in \mathbb{T}^d \times C$;
- (b) $\partial_x \tilde{\xi}_{\beta, \theta}(x) \geq \exp(2b(1 - \exp[-b/(2\rho_D)])/\rho_D)$ for $(\theta, x) \in (\mathbb{T}^d \setminus \mathcal{I}_0) \times E \cap \tilde{\Xi}_\beta^{-1}(\mathbb{T}^d \times E)$;
- (c) $\exp(-2b(1+c)/\rho_D) < \partial_x \tilde{\xi}_{\beta, \theta}(x) \leq \exp(2b/\rho_D)$ for all $(\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma)$.

Proof. Note that due to Proposition 4.1(a), we have that $(\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma)$ necessarily implies $\xi_\beta(t, \theta, x) \in [-1, 1+c]$ for all $t \in (0, 1/\rho_D]$.

Now, (c) follows from equation (3.3) since we have $2b \geq \partial_x F_\beta > -2b(1+c)$ on $\mathbb{T}^d \times [-1, 1+c]$. Similarly, we obviously get item (b) as long as $\xi_\beta(t, \theta, x) \in E$ for all $t \in [0, 1/\rho_D]$ which necessarily holds for $(\theta, x) \in (\mathbb{T}^d \setminus \mathcal{I}_0) \times E \cap \tilde{\Xi}_\beta^{-1}(\mathbb{T}^d \times E)$ due to Proposition 4.1 (b).

It remains to consider (a). Note that for all $x \in C$, all $\beta \in [0, 1]$, each $\theta \in \mathbb{T}^d$, and $t \in [0, 1/\rho_D - \delta_1]$ we have $\xi_\beta(t, \theta, x) \geq \xi_\beta(t, \theta, 1-c) \geq 1-c$. Suppose for a contradiction there were $\theta' \in \mathbb{T}^d$ and $\beta \in [0, \beta_+]$ such that $\tilde{\xi}_{\beta, \theta'}(1-c) = -2$. Note that in this case $\xi_\beta(t, \theta', 1-c) \geq -2$ holds necessarily for all $t \in [0, 1/\rho_D]$ because of Proposition 4.1(a). Thus, (3.3) yields

$$\begin{aligned} \partial_x \tilde{\xi}_{\beta, \theta'}(x) &= \exp\left(\int_0^{1/\rho_D} \partial_x F_\beta(s\rho + \theta', \xi_\beta(s, \theta', x)) ds\right) \\ &= \exp\left(-2b \int_0^{1/\rho_D - \delta_1} \xi_\beta(s, \theta', x) ds - 2b \int_{1/\rho_D - \delta_1}^{1/\rho_D} \xi_\beta(s, \theta', x) ds\right) \\ &\leq \exp(-2b(1-c)(1/\rho_D - \delta_1) + 4b\delta_1) \end{aligned} \tag{4.1}$$

for all $x \in [1 - c, 1 + c]$. Hence in this case, we had $\tilde{\xi}_{\beta_+, \theta}(1 + c) \leq \tilde{\xi}_{\beta, \theta}(1 + c) \leq -2 + 2c \cdot \exp(-2b(1 - c)(1/\rho_D - \delta_1) + 4b\delta_1) < -1$ (where the last inequality holds for large enough b if we assume $\delta_1 < 1/(36\rho_D)$) contradicting the definition of β_+ . Thus, we have $\tilde{\xi}_{\beta, \theta}(x) > -2$ for all $\theta \in \mathbb{T}^d$, $x \in C$ and $\beta \in [0, \beta_+]$ and hence (4.1) yields an upper bound for $\partial_x \tilde{\xi}_{\beta, \theta}(x)$ with $(\theta, x) \in \mathbb{T}^d \times C$. \square

The core part of this article is to show that there is $\mathcal{J}_{0, \beta}$ and $s > 0$ such that (A6) and (A7) hold. The respective proof is given with the next lemma which sheds light on the mechanism by which the sensitive interplay of contraction and expansion is realised. In short words, the problem is to seize control over the second derivatives of solutions of an ode by means of its right-hand side.

Lemma 4.3. *Suppose F_β is given by (*), R is small enough, b is sufficiently large, and $\beta \in [\beta_-, \beta_+]$. Then there is $\mathcal{J}_{0, \beta} \subseteq \mathcal{I}_0$ such that $\partial_\theta^2 \tilde{\xi}_{\beta, \theta}(x) > \exp(b\delta_2/4)$ for arbitrary $x \in C$, $\theta \in \mathbb{S}^{d-1}$, and $\theta \in \mathcal{J}_{0, \beta}$. Further, $\mathcal{J}_{0, \beta}$ contains all $\theta \in \mathbb{T}^d$ with $\tilde{\xi}_{\beta, \theta}(1 - c) \leq -1 + \exp(-b/(2\rho_D))$ and (A6) is satisfied.*

For later use, we provide some crude and straightforward estimates in the following auxiliary statement. We denote by $\mathbf{1}_A$ the characteristic function of a set $A \subseteq \mathbb{T}^D$, that is, $\mathbf{1}_A = 1$ on A and $\mathbf{1}_A = 0$ on $\mathbb{T}^D \setminus A$.

Claim 4.3.1. *For $(\theta, x) \in \mathbb{T}^d \times C$, $\beta \in [0, \beta_+]$, $t_1 \in [0, 1/\rho_D - \delta_1]$, and $t \in [t_1, 1/\rho_D]$, we have under the assumption of sufficiently large b that*

$$\int_0^{t-t_1} \exp\left(\int_s^{t-t_1} \partial_x F_\beta((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)) d\tau\right) ds \leq \exp(5b\delta_1). \quad (4.2)$$

Further, suppose $\tilde{\xi}_{\beta, \theta}(x) \leq -3/4$ and $t \geq 1/\rho_D - \delta_2/2$. There is $R_0 < R$ such that for sufficiently large b

$$\int_0^{t-t_1} \mathbf{1}_{B_{R_0}(\bar{\theta})}((s + t_1)\rho + \theta) \exp\left(\int_s^{t-t_1} \partial_x F_\beta((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)) d\tau\right) ds \geq \exp(b\delta_2/2). \quad (4.3)$$

Finally, if $0 \leq t_1 \leq 1/\rho_D - 5\delta_1$ and $t \in [t_1, 1/\rho_D]$, then we have for all $(\theta, x) \in \mathbb{T}^d \times C$ and sufficiently large b that

$$\exp\left(\int_0^{t-t_1} \partial_x F_\beta((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)) d\tau\right) \leq 1. \quad (4.4)$$

Proof of the claim. The relations can be seen in a similar fashion as (4.1). In particular, we make use of the fact that $\xi_\beta(\tau + t_1, \theta, x) \geq -2$ for all $\tau \in [0, 1/\rho_D - t_1] \supseteq [0, t - t_1]$ and $\xi_\beta(\tau + t_1, \theta, x) \geq 1 - c$ for $\tau \leq 1/\rho_D - \delta_1 - t_1$. For $x \in C$, this implies

$$\exp\left(\int_s^{t-t_1} \partial_x F_\beta((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)) d\tau\right) = \exp\left(-2b \int_s^{t-t_1} \xi_\beta(\tau + t_1, \theta, x) d\tau\right) \leq \exp(4b\delta_1)$$

such that

$$\int_0^{t-t_1} \exp\left(\int_s^{t-t_1} \partial_x F_\beta((\tau + t_1)\rho + \theta, \xi_\beta(\tau + t_1, \theta, x)) d\tau\right) ds \leq 1/\rho_D \cdot \exp(4b\delta_1),$$

which is smaller than $\exp(5b\delta_1)$ for big enough b .

For the second inequality, note that there is $0 < \tilde{R} < R$ such that for big enough b we have $F_\beta(\theta, -3/4) \geq 0$ for all $\theta \notin B_{\tilde{R}}(\bar{\theta})$ and all β . Hence, for all θ and x as in the assumptions, we

necessarily have that $\xi_\beta(\tau, \theta, x) \leq -3/4$ for all $\tau \in [0, 1/\rho_D]$ with $[\theta + \tau\rho, \theta + \omega] \cap B_{\tilde{R}}(\bar{\theta}) = \emptyset$. Set $R_0 = (R + \tilde{R})/2$. Then,

$$\begin{aligned} & \int_0^{t-t_1} \mathbf{1}_{B_{R_0}(\bar{\theta})}((s+t_1)\rho + \theta) \exp\left(\int_s^{t-t_1} \partial_x F_\beta((\tau+t_1)\rho + \theta, \xi_\beta(\tau+t_1, \theta, x)) d\tau\right) ds \\ & \geq \int_0^{t-t_1} \mathbf{1}_{B_{R_0}(\bar{\theta}) \setminus B_{\tilde{R}}(\bar{\theta})}((s+t_1)\rho + \theta) \exp\left(\int_s^{t-t_1} \partial_x F_\beta((\tau+t_1)\rho + \theta, \xi_\beta(\tau+t_1, \theta, x)) d\tau\right) ds \\ & \geq (R_0 - \tilde{R})/\rho_D \cdot \exp\left(-2b \int_{1/\rho_D - \delta_2 - t_1}^{t-t_1} \xi_\beta(\tau+t_1, \theta, x) d\tau\right) \geq (R_0 - \tilde{R})/\rho_D \cdot \exp(3/4 \cdot b\delta_2) \end{aligned}$$

which is clearly bigger than $\exp(b\delta_2/2)$ for large enough b .

For the last relation, note that since $t_1 \leq 1/\rho_D - 5\delta_1$, we have for $t \geq 1/\rho_D - \delta_1$ that

$$\begin{aligned} & \int_0^{t-t_1} \partial_x F_\beta((\tau+t_1)\rho + \theta, \xi_\beta(\tau+t_1, \theta, x)) d\tau \\ & = \int_0^{1/\rho_D - \delta_1 - t_1} \partial_x F_\beta((\tau+t_1)\rho + \theta, \xi_\beta(\tau+t_1, \theta, x)) d\tau \\ & \quad + \int_{1/\rho_D - \delta_1 - t_1}^{t-t_1} \partial_x F_\beta((\tau+t_1)\rho + \theta, \xi_\beta(\tau+t_1, \theta, x)) d\tau \leq -8b\delta_1 \cdot (1-c) + 4b\delta_1 < 0 \end{aligned}$$

since $c < 1/4$. Note that if $t < 1/\rho_D - \delta_1$, then (4.4) is obvious. \square

Proof of Lemma 4.3. Let us fix some notation. For the rest of this proof, ϑ is always assumed to be some element of \mathbb{R}^d (the tangent space of \mathbb{T}^d at any $\theta \in \mathbb{T}^d$) with $|\vartheta| = 1$. In contrast, Δ and Δ' always denote elements of $\mathbb{S}^d \subseteq \mathbb{R}^D$ in the orthogonal complement of ρ in the following. Set $T_\tau = \{\theta_0 + \tau\rho + \varepsilon\Delta : \Delta \perp \rho, |\Delta| = 1 \text{ and } |\varepsilon| \leq R\}$ where $\tau \in (0, 1/\rho_D)$, that is, T_τ is a d -dimensional disk of radius R orthogonal to ρ , centred at $\theta_0 + \tau\rho$. Similarly, set $\tilde{T}_\tau = \{\theta_0 + \tau\rho + \varepsilon\Delta : \Delta \perp \rho, |\Delta| = 1 \text{ and } |\varepsilon| \leq r_b\}$ where $r_b = \exp(-9b\delta_1)$. We denote by $P\theta$ the *orthogonal projection* of $\theta \in \mathbb{T}^D$ onto $[\theta_0, \theta_0 + \omega]$ so that $\theta = P\theta + \varepsilon\Delta$, where $|\varepsilon|$ is minimal (with $\Delta \perp \rho$ and $|\Delta| = 1$ as above).⁷ Set $t_1 = 1/(4\rho_D)$ and note that—if R is small enough— $T_{t_1} \cap \mathbb{T}^d = \emptyset$ and $[I_0, T_{t_1}] \cap B_{\tilde{R}}(\bar{\theta}) = \emptyset$. Let t_2 be such that T_{t_2} has a positive distance to $B_{\tilde{R}}(\bar{\theta})$ and such that $T_{t_2} \cap \mathbb{T}^d = \emptyset$ and $[I_0, T_{t_2}] \cap B_{\tilde{R}}(\bar{\theta}) = B_{\tilde{R}}(\bar{\theta})$. Again, we might have to assume small enough R in order for such t_2 to exist. Finally, $t_3 > t_2$ with $\tilde{T}_{t_3} \cap \mathbb{T}^d = \emptyset$ will be chosen to be close to $1/\rho_D$ so that within one iteration, orbits starting in $\tilde{I}_0 = \{\theta \in \mathbb{T}^d : [\theta, \theta + \omega] \cap \tilde{T}_{t_1} \neq \emptyset\}$ enter and leave the bump between \tilde{T}_{t_1} and \tilde{T}_{t_3} while the remaining time between \tilde{T}_{t_3} and \mathbb{T}^d will be negligibly short. We let $t_i(\theta) \in [0, 1/\rho_D]$ be such that $\theta + t_i(\theta)\rho \in T_{t_i}$ for $i = 1, 2$ and $\theta \in I_0$. Likewise for $\theta \in \tilde{I}_0$, we denote by $t_3(\theta) \in [0, 1/\rho_D]$ that time for which $\theta + t_3(\theta)\rho \in \tilde{T}_{t_3}$. By considering small enough R , we may assume without loss of generality that $t_2(\theta) > 1/\rho_D - \delta_2/2$ for each $\theta \in \tilde{I}_0$. Note that the t_i 's are (restrictions of) affine linear maps whose derivatives are given by a constant matrix whose norm we denote by κ for the rest of this proof (note that obviously $dt_1(\theta) = dt_2(\theta) = dt_3(\theta)$, where d denotes the total derivative).

The Hessian of $g(\theta) = h(|\theta - \bar{\theta}|)$ is easily seen to be

$$d^2 g(\theta) = d\left(\frac{h'(|\theta - \bar{\theta}|)}{|\theta - \bar{\theta}|}(\theta - \bar{\theta})\right) = \left(\frac{h''(|\theta - \bar{\theta}|)}{|\theta - \bar{\theta}|^2} - \frac{h'(|\theta - \bar{\theta}|)}{|\theta - \bar{\theta}|^3}\right)(\theta - \bar{\theta}) \cdot (\theta - \bar{\theta})^\top + \frac{h'(|\theta - \bar{\theta}|)}{|\theta - \bar{\theta}|} I_D,$$

⁷In the following, we only consider $P\theta$ for θ close to $[\theta_0, \theta_0 + \omega]$ so that we do not have to worry about the well-definition of P .

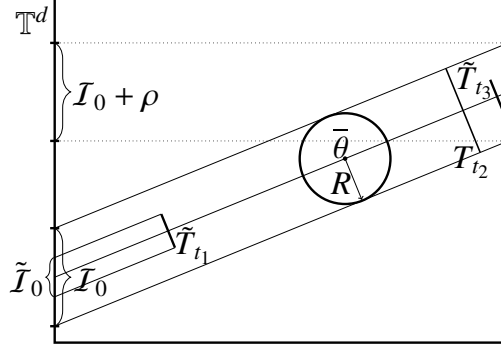


Figure 2: The base space for $D = 2$. We subdivide one iteration into three subsequent iterations: first, from $\tilde{I}_0 \subseteq \mathbb{T}^d$ to \tilde{T}_{t_1} . Then, further to \tilde{T}_{t_3} . Finally, from \tilde{T}_{t_3} to $\tilde{I}_0 + \omega \subseteq \mathbb{T}^d$. If $\theta_0 + \tau\rho \in B_R(\bar{\theta})$, the disks \tilde{T}_τ are sections of the tangents of the level sets of g at $\theta_0 + \tau\rho$.

where I_D denotes the D -dimensional unit matrix. Hence for $\theta = P\theta + \varepsilon\Delta'$, we have

$$\partial_{\Delta} g(\theta) = \varepsilon \cdot \frac{h'(|\theta - \bar{\theta}|)}{|\theta - \bar{\theta}|} \langle \Delta', \Delta \rangle \quad \text{and} \quad (4.5)$$

$$\partial_{\Delta}^2 g(\theta) = \varepsilon^2 \cdot \left(\frac{h''(|\theta - \bar{\theta}|)}{|\theta - \bar{\theta}|^2} - \frac{h'(|\theta - \bar{\theta}|)}{|\theta - \bar{\theta}|^3} \right) \langle \Delta', \Delta \rangle^2 + \frac{h'(|\theta - \bar{\theta}|)}{|\theta - \bar{\theta}|}, \quad (4.6)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbb{R}^D .

Having in mind (3.7), we see that in order to show the positivity of the second derivatives of ξ_β with respect to $\vartheta \in \mathbb{S}^{d-1}$, we need small enough upper bounds on the respective first derivatives in order to ensure that the leading term under the integral is the one containing $\partial_\vartheta^2 F_\beta$. To that end, we divide the iteration of an orbit starting at $(\theta, x) \in \tilde{I}_0 \times C$ into three time intervals (see Figure 2). Within the first interval, $[0, t_1(\theta)]$, variation with respect to θ only occurs due to the θ -dependence of $t_1(\theta)$ which turns out to be negligible. The last time interval, $[t_3(\theta), 1/\rho_D]$, will turn out to be negligible as we can assume its length to be small. For the intermediate time interval $[t_1(\theta), t_3(\theta)]$, the crucial point is that by the choice of the sets \tilde{T}_τ perpendicular to ρ and hence parallel to the level sets of g at the point $\theta_0 + \tau\rho$, the derivatives with respect to ϑ become derivatives with respect to some $\Delta \perp \rho$. By (4.5), this implies that in a distance $\varepsilon = r_b$ of $\theta_0 + \tau\rho$ (where $\tau \in [t_1(\theta), t_3(\theta)]$), these derivatives are exponentially small in b (recall that $r_b = \exp[-9b\delta_1]$).

While the first derivatives with respect to ϑ are thus negligible, we will show in Claim 4.3.3 that $(\partial_\Delta^2 \xi_\beta)(\tau - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x))$ is bounded away from 0 for each $\tau \in [t_2, t_3]$, provided $\xi_\beta(1/\rho_D, \theta, x)$ is not too far from E . In conclusion, we will show that $\partial_\vartheta^2 \xi_\beta(1/\rho_D, \theta, x)$ is bounded away from 0. Together with the next claim, this will finish the proof of Lemma 4.3.

Claim 4.3.2. *Under the assumption of Lemma 4.3, suppose there is $c_0 > 0$ (independent of b) such that $(\partial_\Delta^2 \xi_\beta)(t_2 - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)) > c_0$ for all $\theta \in \tilde{I}_0$ and $x \in C$ with $\tilde{\xi}_{\beta, \theta}(x) \leq -3/4$.*

Then there is a closed and convex set $\mathcal{J}_{0, \beta} \subseteq \tilde{I}_0$ such that $\tilde{\xi}_{\beta, \theta}(1 - c) > -1 + \exp(-b/(2\rho_D))$ if $\theta \notin \mathcal{J}_{0, \beta}$. Further, $\tilde{\xi}_{\beta, \theta}(x) \leq -3/4$ for each $\theta \in \mathcal{J}_{0, \beta}$ and $x \in C$, and $\mathcal{J}_{0, \beta} \subseteq \mathcal{J}_{0, \beta'}$ for $\beta \leq \beta' \in [\beta_-, \beta_+]$.

Proof of the claim. For the rest of this proof, given $\theta \in T_{t_2}$, we denote by θ' that point in I_0 for which $\theta = \theta' + t_2(\theta')\rho$.

The map $u: T_{t_2} \ni \theta \mapsto \xi_\beta(t_2(\theta'), \theta', 1)$ —we keep the dependence of u on β implicit—assumes its minimum in $\theta_0 + t_2\rho$ and moreover satisfies

$$u(\theta) = \hat{u}(|\theta - (\theta_0 + t_2\rho)|), \quad (4.7)$$

where $\hat{u}: [0, R] \rightarrow X$ is some non-decreasing function. This can be seen as follows: First, we see that $\xi_\beta(t_1(\theta'), \theta', 1) = 1$ for each $\theta' \in \mathcal{I}_0$ since $F_\beta(\theta' + \tau\rho, 1) = 0$ for all $\tau \in [0, t_1(\theta')]$ by definition of t_1 . Hence, $u(\theta) = \xi_\beta(t_2 - t_1, \theta - (t_2 - t_1)\rho, 1)$. Now note that for $\tau \in [0, t_2 - t_1]$, we have

$$\begin{aligned} |\theta - (t_2 - t_1)\rho + \tau\rho - \bar{\theta}|^2 &= |\theta - (t_2 - t_1)\rho + \tau\rho - (\theta_0 + (t_1 + \tau)\rho)|^2 + |\theta_0 + (t_1 + \tau)\rho - \bar{\theta}|^2 \\ &= |\theta - (\theta_0 + t_2\rho)|^2 + |\theta_0 + (t_1 + \tau)\rho - \bar{\theta}|^2. \end{aligned}$$

Since $g(\cdot) = h(|(\cdot) - \bar{\theta}|)$, we therefore have that there is $\hat{F}: T_{t_2} \times [0, t_2 - t_1] \times X \rightarrow \mathbb{R}$ with $F_\beta(\theta - (t_2 - t_1)\rho + \tau\rho, x) = \hat{F}(|\theta - (\theta_0 + t_2\rho)|, \tau, x)$, where \hat{F} is non-decreasing in the first coordinate. This proves (4.7).

Set

$$\mathcal{J}_{0,\beta} = \{\theta' \in \mathcal{I}_0 : u(\theta) \leq -1 + \exp(-b/(2\rho_D)) + 1/2 \cdot \exp(-b/\rho_D)\}.$$

Obviously, $\mathcal{J}_{0,\beta}$ is closed and $\mathcal{J}_{0,\beta} \subseteq \mathcal{J}_{0,\beta'}$ for $\beta' \geq \beta$. The convexity of $\mathcal{J}_{0,\beta}$ follows from (4.7). It hence remains to show that for sufficiently large b we have $\mathcal{J}_{0,\beta} \subseteq \tilde{\mathcal{I}}_0$, $\tilde{\xi}_{\beta,\theta}(1 - c) > -1 + \exp(-b/(2\rho_D))$ for $\theta' \notin \mathcal{J}_{0,\beta}$, and $\tilde{\xi}_{\beta,\theta'}(x) \leq -3/4$ for all $\theta' \in \mathcal{J}_{0,\beta}$ and $x \in C$.

First, we show $\tilde{\xi}_{\beta,\theta}(1 - c) > -1 + \exp(-b/(2\rho_D))$ for $\theta' \notin \mathcal{J}_{0,\beta}$. Obviously,

$$\xi_\beta(1/\rho_D, \theta', 1 - c) \leq -1 + \exp[-b/(2\rho_D)]$$

if and only if

$$\xi_\beta(t_2(\theta'), \theta', 1 - c) \leq \xi_\beta^-(1/\rho_D - t_2(\theta'), \theta' + \omega, -1 + \exp[-b/(2\rho_D)])$$

where $\xi_\beta^-(t, \theta', x) = \xi_\beta(-t, \theta', x)$. Similarly to Proposition 4.2 (a), we get for each $x \in C$ that $\partial_x \xi_\beta(t_2(\theta'), \theta', x) \leq \exp(-2b(1 - c)(1/\rho_D - \delta_1) + 4b\delta_1)$, which is smaller than $\exp(-b/\rho_D)$, since $\delta_1 < 1/(36\rho_D)$ and $c < 1/4$. Therefore,

$$|\xi_\beta(t_2(\theta'), \theta', 1 + c) - \xi_\beta(t_2(\theta'), \theta', 1 - c)| \leq 2c \exp(-b/\rho_D) \leq 1/2 \exp(-b/\rho_D). \quad (4.8)$$

In particular, this implies $|u(\theta) - \xi_\beta(t_2(\theta'), \theta', 1 - c)| < 1/2 \cdot \exp(-b/\rho_D)$. As further $\xi_\beta^-(1/\rho_D - t_2(\theta'), \theta' + \omega, -1 + \exp[-b/(2\rho_D)]) \leq -1 + \exp[-b/(2\rho_D)]$ (due to Proposition 4.1 (b)), this yields that $\xi_\beta(t_2(\theta'), \theta', 1 - c) > \xi_\beta^-(1/\rho_D - t_2(\theta'), \theta' + \omega, -1 + \exp[-b/(2\rho_D)])$ if

$$u(\theta) > -1 + \exp(-b/(2\rho_D)) + 1/2 \exp(-b/\rho_D). \quad (4.9)$$

Hence, $\tilde{\xi}_{\beta,\theta}(1 - c) > -1 + \exp(-b/(2\rho_D))$ for $\theta' \notin \mathcal{J}_{0,\beta}$.

Given $\theta' \in \mathcal{I}_0$ with $u(\theta) = \xi_\beta(t_2(\theta'), \theta', 1) \geq -1$, there is $y \in [-1, u(\theta)]$ such that

$$\begin{aligned} \tilde{\xi}_{\beta,\theta'}(1) &= \xi_\beta(1/\rho_D - t_2(\theta'), \theta, -1) + \partial_x \xi_\beta(1/\rho_D - t_2(\theta'), \theta, y) \cdot |-1 - u(\theta)| \\ &\leq -1 + \exp(b\delta_2) \cdot |-1 - u(\theta)| \end{aligned}$$

where we used (3.3) (recall that $t_2(\theta) > 1/\rho_D - \delta_2/2$) and the fact that $\xi_\beta(1/\rho_D - t_2(\theta'), \theta, -1) = -1$. Thus, $\tilde{\xi}_{\beta,\theta'}(1) > -7/8$ necessarily means $u(\theta) \geq -1 + 1/8 \exp(-b\delta_2)$ which is bigger than the right-hand side of (4.9) for large enough b as $\delta_2 < \delta_1 \leq 1/(36\rho_D)$. Hence, $\tilde{\xi}_{\beta,\theta'}(x) \leq -3/4$ for all $\theta' \in \mathcal{J}_{0,\beta}$ and $x \in C$.

We are left to show that $\mathcal{J}_{0,\beta} \subseteq \tilde{\mathcal{I}}_{0,\beta}$, which is equivalent to showing that (4.9) holds for each $\theta \in T_{t_2} \setminus \tilde{T}_{t_2}$. By the above, we may assume without loss of generality that $\tilde{\xi}_{\beta,\theta'}(1) \leq -3/4$ for all $\theta' \in \tilde{\mathcal{I}}_0$ so that $(\partial_{\Delta}^2 \xi_{\beta})(t_2 - t_1, \theta' + t_1(\theta')\rho, 1) > c_0$ by the hypothesis of this claim. Note that by definition of β_+ and due to Proposition 4.1 (a), it follows from (4.8) that $u(\theta) \geq u(\theta_0 + t_2\rho) \geq -1 - 1/2 \exp(-b/\rho_D)$. Hence, for θ on the boundary of \tilde{T}_{t_2} , we get by means of the lower bound c_0 on the second derivatives that

$$\begin{aligned} u(\theta) &\geq u(\theta_0 + t_2\rho) + c_0 \cdot |\theta - (\theta_0 + t_2\rho)|^2 \geq -1 - 1/2 \exp(-b/\rho_D) + c_0 r_b^2 \\ &= -1 - 1/2 \exp(-b/\rho_D) + c_0 \exp(-18b\delta_1) \\ &> -1 + \exp[-b/(2\rho_D)] + 1/2 \exp(-b/\rho_D) \end{aligned}$$

for large enough b as $\delta_1 < 1/(36\rho_D)$. By means of (4.7), this proves (4.9) for all $\theta \in T_{t_2} \setminus \tilde{T}_{t_2}$. \square

It remains to compute upper bounds on the first derivatives $\partial_{\theta}\xi_{\beta}$ and lower bounds for the second derivatives $\partial_{\theta}^2\xi_{\beta}$. For $\theta \in \tilde{\mathcal{I}}_0$ and $x \in C$, we have

$$\begin{aligned} |\partial_{\theta}\xi_{\beta}(t_1(\theta), \theta, x)| &\leq |(\partial_{\theta}\xi_{\beta})(t_1(\theta), \theta, x)| + |\partial_t\xi_{\beta}(t_1(\theta), \theta, x) \cdot \partial_{\theta}t_1(\theta)| \\ &\leq \kappa(1+c)^2b. \end{aligned} \quad (4.10)$$

This is due to the fact that $(\partial_{\theta}\xi_{\beta})(t_1(\theta), \theta, x) = 0$ (see (3.4) and recall that $[\tilde{\mathcal{I}}_0, \tilde{T}_{t_1}] \cap B_R(\bar{\theta}) = \emptyset$) and because $\xi_{\beta}(t_1(\theta), \theta, x) \in C$ for all $(\theta, x) \in \mathbb{T}^d \times C$ such that

$$|\partial_t\xi_{\beta}(t_1(\theta), \theta, x)| = |F_{\beta}(\theta + t_1(\theta)\rho, \xi_{\beta}[t_1(\theta), \theta, x])| \leq (1+c)^2b. \quad (4.11)$$

For $t \in [t_1, t_3]$, $\theta \in \tilde{\mathcal{I}}_0$ and $x \in C$, we further have

$$\begin{aligned} &\left| (\partial_{\Delta}\xi_{\beta})(t - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x)) \right| \\ &\leq \int_0^{t-t_1} \left| (\partial_{\Delta}F_{\beta})(\theta + [s + t_1(\theta)]\rho, \xi_{\beta}[s + t_1(\theta), \theta, x]) \right| \\ &\quad \cdot \exp\left(\int_s^{t-t_1} (\partial_x F_{\beta})(\theta + [\tau + t_1(\theta)]\rho, \xi_{\beta}[\tau + t_1(\theta), \theta, x]) d\tau \right) ds \\ &\leq \iota \cdot h''(0) \cdot b/(1 - b^{-1/2})r_b \\ &\quad \cdot \int_0^{t-t_1} \exp\left(\int_s^{t-t_1} (\partial_x F_{\beta})(\theta + [\tau + t_1(\theta)]\rho, \xi_{\beta}[\tau + t_1(\theta), \theta, x]) d\tau \right) ds \\ &\leq \iota \cdot h''(0) \cdot b/(1 - b^{-1/2}) \exp(5b\delta_1)r_b \leq \exp(6b\delta_1)r_b \end{aligned} \quad (4.12)$$

for sufficiently large b , where we used (4.5) in the second step (with ι such that $|h'(y)/y| \leq \iota|h''(0)|$ for all $y \geq 0$) and (4.2) in the second to the last step. Observe that (4.12) is an upper bound on $|(\partial_{\Delta}\xi_{\beta})(t - t_1, \theta + t_1(\theta)\rho, x)|$ for all $\Delta \perp \rho$ of length 1.

Now, the derivative of the map $\tilde{\mathcal{I}}_0 \times C \ni (\theta, x) \mapsto \xi_{\beta}(t - t_1 + t_1(\theta), \theta, x)$ in direction of an arbitrary $\vartheta \in \mathbb{R}^d$ with $|\vartheta| = 1$ is given by

$$\begin{aligned} \partial_{\theta}\xi_{\beta}(t - t_1 + t_1(\theta), \theta, x) &= \partial_{\theta}\xi_{\beta}(t - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x)) \\ &= (d_{\theta}\xi_{\beta})(t - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x)) \cdot (\vartheta + \partial_{\theta}t_1(\theta)\rho) \\ &\quad + (\partial_x\xi_{\beta})(t - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x)) \cdot \partial_{\theta}\xi_{\beta}(t_1(\theta), \theta, x) \\ &= |\vartheta + \partial_{\theta}t_1(\theta)\rho| \cdot (\partial_{\Delta}\xi_{\beta})(t - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x)) \\ &\quad + (\partial_x\xi_{\beta})(t - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x)) \cdot \partial_{\theta}\xi_{\beta}(t_1(\theta), \theta, x) \end{aligned} \quad (4.13)$$

where $(d_\theta \xi_\beta)(t, \theta, x)$ denotes the total derivative of the map $\theta \mapsto \xi_\beta(t, \theta, x)$ (for fixed t and x) and $\Delta = (\vartheta + \partial_\vartheta t_1(\theta)\rho)/|\vartheta + \partial_\vartheta t_1(\theta)\rho|$ is indeed orthogonal to ρ . Note that due to (4.4), $(\partial_x \xi_\beta)(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta[t_1(\theta), \theta, x]) \leq 1$ for all $t \in [t_1, 1/\rho_D]$ since $t_1(\theta) \leq t_1 + R < t_1 + \delta_1 < 1/\rho_D - 5\delta_1$ (recall that $t_1 = 1/(4\rho_D)$). By means of (4.10) and (4.12), we hence have

$$|\partial_\theta \xi_\beta(t - t_1 + t_1(\theta), \theta, x)| \leq (1 + \kappa|\rho|) \exp(6b\delta_1)r_b + \kappa(1 + c)^2b. \quad (4.14)$$

We thus have upper bounds on the first derivatives of ξ_β with respect to Δ and ϑ . We proceed with the second derivatives.

Claim 4.3.3. $(\partial_\Delta^2 \xi_\beta)(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)) > \exp(b\delta_2/2)$ for all $\theta \in \tilde{I}_0$, $t \in [t_2, t_3]$, and $x \in C$ with $\tilde{\xi}_{\beta, \theta}(x) \leq -3/4$.

Proof of the claim. As $h' \upharpoonright_{(0, R)} < 0$ and $\partial_\Delta^2 g(\bar{\theta}) < 0$, we see by means of (4.6) that there is $\gamma_1 > 0$ such that for sufficiently large b we have $\partial_\Delta^2 g < -\gamma_1$ on $B_{R_0}(\bar{\theta}) \cap [\tilde{I}_0, \tilde{I}_0 + \omega]$, where $R_0 > 0$ is as in Claim 4.3.1. Let

$$\gamma_2 = \max_{\theta \in \mathbb{T}^D \setminus B_{R_0}(\bar{\theta})} h''(|\theta - \bar{\theta}|)/|\theta - \bar{\theta}|^2 - h'(|\theta - \bar{\theta}|)/|\theta - \bar{\theta}|^3 \geq 0$$

and observe—again by means of (4.6)—that $\partial_\Delta^2 g \leq \gamma_2 r_b^2$ on $(B_R(\bar{\theta}) \setminus B_{R_0}(\bar{\theta})) \cap [\tilde{I}_0, \tilde{I}_0 + \omega]$.

For θ , x , and t as in the hypothesis, we thus have

$$\begin{aligned} & \int_0^{t-t_1} (\partial_\Delta^2 g)(\theta + [s + t_1(\theta)]\rho) \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]) d\tau\right) ds \\ & \leq \int_0^{t-t_1} \left(\gamma_2 r_b^2 \mathbf{1}_{B_R(\bar{\theta}) \setminus B_{R_0}(\bar{\theta})}(\bar{\theta})(\theta + [s + t_1(\theta)]\rho) - \gamma_1 \mathbf{1}_{B_{R_0}(\bar{\theta})}(\bar{\theta})(\theta + [s + t_1(\theta)]\rho) \right) \\ & \quad \cdot \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]) d\tau\right) ds \\ & \leq \gamma_2 r_b^2 \exp(5b\delta_1) - \gamma_1 \exp(b\delta_2/2) \leq -\gamma_3 \exp(b\delta_2/2) \end{aligned} \quad (4.15)$$

for some $\gamma_3 > 0$, where we used (4.2) and (4.3) in the second to last step (recall that $r_b = \exp(-9b\delta_1)$).

Now, plugging (4.12) and (4.2) into (3.7) (observe that the term with the mixed derivatives of F_β vanishes for $(*)$) yields for each $t \in [t_2, t_3]$ that

$$\begin{aligned} & \left| (\partial_\Delta^2 \xi_\beta)(t - t_1, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)) \right| \\ & = \left| \int_0^{t-t_1} \left[(\partial_x^2 F_\beta)(\theta + [s + t_1(\theta)]\rho, \xi_\beta[s + t_1(\theta), \theta, x]) \left((\partial_\Delta \xi_\beta)[s, \theta + t_1(\theta)\rho, \xi_\beta(t_1(\theta), \theta, x)] \right)^2 \right. \right. \\ & \quad \left. \left. + (\partial_\Delta^2 F_\beta)(\theta + [s + t_1(\theta)]\rho, \xi_\beta[s + t_1(\theta), \theta, x]) \right] \right. \\ & \quad \left. \cdot \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]) d\tau\right) ds \right| \\ & \geq -2b \exp(17b\delta_1)r_b^2 - \beta b/(1 - b^{-1/2}) \\ & \quad \cdot \int_0^{t-t_1} (\partial_\Delta^2 g)(\theta + [s + t_1(\theta)]\rho) \exp\left(\int_s^{t-t_1} (\partial_x F_\beta)(\theta + [\tau + t_1(\theta)]\rho, \xi_\beta[\tau + t_1(\theta), \theta, x]) d\tau\right) ds \\ & \geq -2b \exp(17b\delta_1)r_b^2 + \gamma_3 \beta b/(1 - b^{-1/2}) \exp(b\delta_2/2) \end{aligned}$$

which is bigger than $\exp(b\delta_2/2)$ for sufficiently large b , where we used (4.15) in the last step. \square

Thus, the assumptions of Claim 4.3.2 are met and it remains to show that $\partial_{\vartheta}^2 \tilde{\xi}_{\beta, \theta}(x) > \exp(b\delta_2/4)$ for $x \in C$, $\vartheta \in \mathbb{S}^{d-1}$, and $\theta \in \mathcal{J}_{0\beta}$. Plugging (4.4) into (3.5), yields

$$\left| \left(\partial_x^2 \xi_{\beta} \right) \left(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}[t_1(\theta), \theta, x] \right) \right| \leq 2b/\rho_D.$$

Analogously, with (3.6) and (4.12) we get

$$\left| \left(\partial_{\Delta} \partial_x \xi_{\beta} \right) \left(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}[t_1(\theta), \theta, x] \right) \right| \leq 2b/\rho_D \cdot \exp(6b\delta_1)r_b.$$

Finally, note that

$$\partial_{\vartheta}^2 \xi_{\beta}(t_1(\theta), \theta, x) = \left(\partial_{\vartheta}^2 \xi_{\beta} \right) (t_1(\theta), \theta, x) + 2 \left(\partial_t \partial_{\vartheta} \xi_{\beta} \right) (t_1(\theta), \theta, x) \cdot \partial_{\vartheta} t_1(\theta) + \partial_t^2 \xi_{\beta}(t_1(\theta), \theta, x) \cdot (\partial_{\vartheta} t_1(\theta))^2, \quad (4.16)$$

where we used the fact that $\partial_{\vartheta}^2 t_1(\theta) = 0$. By means of (3.4), we have that $\partial_{\vartheta} \xi_{\beta}(\tau, \theta, x) = 0$ for all $\tau \in [0, 1/\rho_D - \delta_1]$ so that both $\left(\partial_{\vartheta}^2 \xi_{\beta} \right) (t_1(\theta), \theta, x)$ and $\left(\partial_t \partial_{\vartheta} \xi_{\beta} \right) (t_1(\theta), \theta, x) \cdot \partial_{\vartheta} t_1(\theta)$ vanish. Further,

$$\begin{aligned} \partial_t^2 \xi_{\beta}(t_1(\theta), \theta, x) &= \partial_x F_{\beta}(\theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x)) \partial_t \xi_{\beta}(t_1(\theta), \theta, x) \\ &= -2b \cdot \xi_{\beta}(t_1(\theta), \theta, x) \cdot \partial_t \xi_{\beta}(t_1(\theta), \theta, x) \end{aligned} \quad (4.17)$$

where we used that $d_{\theta} F_{\beta}(\theta + t\rho, x) = 0$ for all $t \in [0, 1/\rho_D - \delta_1]$ in the first step. Since $\xi_{\beta}(t_1(\theta), \theta, x) \leq 1 + c$ and due to (4.11), we hence get

$$|\partial_t^2 \xi_{\beta}(t_1(\theta), \theta, x)| \leq 2\kappa^2(1+c)^3 b^2. \quad (4.18)$$

We are now in a position to derive a lower bound on the second derivative of $\tilde{I}_0 \times C \ni (\theta, x) \mapsto \xi_{\beta}(t_3(\theta), \theta, x) = \xi_{\beta}(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}[t_1(\theta), \theta, x])$ in direction of ϑ . From (4.13), we get

$$\begin{aligned} \partial_{\vartheta}^2 \xi_{\beta}(t_3(\theta), \theta, x) &= |\vartheta + \partial_{\vartheta} t_1(\theta)\rho|^2 \cdot \left(\partial_{\Delta}^2 \xi_{\beta} \right) \left(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x) \right) \\ &\quad + 2|\vartheta + \partial_{\vartheta} t_1(\theta)\rho| \left(\partial_{\Delta} \partial_x \xi_{\beta} \right) \left(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x) \right) \cdot \partial_{\vartheta} \xi_{\beta}(t_1(\theta), \theta, x) \\ &\quad + \left(\partial_x^2 \xi_{\beta} \right) \left(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x) \right) \cdot \left(\partial_{\vartheta} \xi_{\beta}(t_1(\theta), \theta, x) \right)^2 \\ &\quad + \left(\partial_x \xi_{\beta} \right) \left(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x) \right) \cdot \partial_{\vartheta}^2 \xi_{\beta}(t_1(\theta), \theta, x). \end{aligned}$$

By the above computations and in particular from Claim 4.3.3, we see that for large enough b the leading term is the one containing $\left(\partial_{\Delta}^2 \xi_{\beta} \right) \left(t_3 - t_1, \theta + t_1(\theta)\rho, \xi_{\beta}(t_1(\theta), \theta, x) \right)$. This yields

$$\left| \partial_{\vartheta}^2 \xi_{\beta}(t_3(\theta), \theta, x) \right| \geq \exp(b\delta_2/3) \quad (4.19)$$

for large enough b . Now, let us consider the derivatives $\partial_{\vartheta}^2 \tilde{\xi}_{\beta, \theta}(x)$. Analogously to (4.13), we get

$$\begin{aligned} \partial_{\vartheta} \tilde{\xi}_{\beta, \theta}(x) &= \partial_{\vartheta} \xi_{\beta} \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_{\beta}(t_3(\theta), \theta, x) \right) \\ &= -\partial_t \xi_{\beta} \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_{\beta}(t_3(\theta), \theta, x) \right) \cdot \partial_{\vartheta} t_3(\theta) \\ &\quad + |\vartheta + \partial_{\vartheta} t_3(\theta)\rho| \cdot \left(\partial_{\Delta} \xi_{\beta} \right) \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_{\beta}(t_3(\theta), \theta, x) \right) \\ &\quad + \left(\partial_x \xi_{\beta} \right) \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_{\beta}(t_3(\theta), \theta, x) \right) \cdot \partial_{\vartheta} \xi_{\beta}(t_3(\theta), \theta, x) \\ &= -\partial_t \xi_{\beta} \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_{\beta}(t_3(\theta), \theta, x) \right) \cdot \partial_{\vartheta} t_3(\theta) \\ &\quad + \left(\partial_x \xi_{\beta} \right) \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_{\beta}(t_3(\theta), \theta, x) \right) \cdot \partial_{\vartheta} \xi_{\beta}(t_3(\theta), \theta, x), \end{aligned}$$

where we used that $F_\beta(\theta + t_3(\theta)\rho + \tau, \cdot) = 0$ for all $\tau \in [0, 1/\rho_D - t_3(\theta)]$ and $\theta \in \tilde{I}_0$ in the last step. By differentiating this expression once more, we straightforwardly obtain

$$\begin{aligned} \partial_\theta^2 \tilde{\xi}_{\beta, \theta}(x) &= \partial_t^2 \xi_\beta \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x) \right) \cdot (\partial_\theta t_3(\theta))^2 \\ &\quad - 2(\partial_t \partial_x \xi_\beta) \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x) \right) \cdot \partial_\theta t_3(\theta) \cdot \partial_\theta \xi_\beta(t_3(\theta), \theta, x) \\ &\quad + \left(\partial_x^2 \xi_\beta \right) \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x) \right) \cdot \left(\partial_\theta \xi_\beta(t_3(\theta), \theta, x) \right)^2 \\ &\quad + \left(\partial_x \xi_\beta \right) \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x) \right) \cdot \partial_\theta^2 \xi_\beta(t_3(\theta), \theta, x). \end{aligned}$$

Let us discuss why $(\partial_x \xi_\beta)(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)) \cdot \partial_\theta^2 \xi_\beta(t_3(\theta), \theta, x)$ is the leading term. To that end, note that since $\xi_\beta(\tau, \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)) < 0$ for all $\tau \in [0, 1/\rho_D - t_3(\theta)]$ and $\theta \in \mathcal{J}_{0, \beta}$, we have $(\partial_x \xi_\beta)(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)) \geq 1$. Together with (4.19), this eventually finishes the proof if we can show that the remaining terms are indeed negligible.

By an analogous computation as in (4.17), we see

$$\begin{aligned} \left| \partial_t^2 \xi_\beta \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x) \right) \right| &\leq 2b \cdot |\xi_\beta(1/\rho_D, \theta, x)| \cdot \partial_t \xi_\beta(1/\rho_D, \theta, x) \\ &\leq 16b^2, \end{aligned}$$

where we used that $\xi_\beta(1/\rho_D, \theta, x) \geq -2$ (see Proposition 4.2 (a)) in the last step. Further,

$$\left(\partial_x \xi_\beta \right) \left(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x) \right) \leq \exp(4b(1/\rho_D - t_3))$$

so that by putting t_3 close enough to $1/\rho_D$ (which is possible if we assume large enough b) we get small enough upper bounds on $(\partial_t \partial_x \xi_\beta)(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x))$ (see (3.1)) as well as $(\partial_x^2 \xi_\beta)(1/\rho_D - t_3(\theta), \theta + t_3(\theta)\rho, \xi_\beta(t_3(\theta), \theta, x)) \cdot (\partial_\theta \xi_\beta(t_3(\theta), \theta, x))^2$ (see (3.5) and (4.14)). \square

There are six more assumptions on $\tilde{\Xi}_\beta$ to be considered. These basically boil down to some weak upper bounds on further derivatives of the first return maps and their inverses.

Let $S > 0$ be such that

$$(\mathcal{A}10) \quad |\partial_\theta \tilde{\xi}_{\beta, \theta}(x)| < S \text{ for all } \theta \in \mathbb{S}^{d-1} \text{ and } (\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma);$$

$$(\mathcal{A}11) \quad |\partial_\theta^2 \tilde{\xi}_{\beta, \theta}(x)| < S^2 \text{ for all } \theta \in \mathbb{S}^{d-1} \text{ and } (\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma);$$

$$(\mathcal{A}12) \quad |\partial_\theta \partial_x \tilde{\xi}_{\beta, \theta}(x)| < \begin{cases} S \alpha_c & \text{for } (\theta, x) \in \mathbb{T}^d \times C \\ S \alpha_u^2 & \text{for } (\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma) \end{cases} \text{ for each } \theta \in \mathbb{S}^{d-1}.$$

Equations (3.4), (3.7) and (3.6) yield that a possible choice to ensure $(\mathcal{A}10)$ – $(\mathcal{A}12)$ for $\tilde{\xi}_{\beta, \theta}$ is to set $S = \exp(9b\delta_1)$. In case of $(\mathcal{A}10)$, this can be seen from

$$\begin{aligned} &|\partial_\theta \tilde{\xi}_{\beta, \theta}(x)| \\ &\stackrel{(3.4)}{\leq} \int_0^{1/\rho_D} \left| (\partial_\theta F_\beta)(s\rho + \theta, \xi_\beta(s, \theta, x)) \right| \exp \left(\int_s^{1/\rho_D} (\partial_x F_\beta)(\tau\rho + \theta, \xi_\beta(\tau, \theta, x)) d\tau \right) ds \\ &\leq \delta_1 b / (1 - b^{-1/2}) \cdot \max_{\theta \in \mathbb{T}^D} |\partial_\theta g(\theta)| \exp(2b\delta_1), \end{aligned}$$

where we used that $\partial_\theta F_\beta$ vanishes for $s < 1/\rho_D - \delta_1$ and that $\xi_\beta(\tau, \theta, x) \geq -1$ for each $\tau \in [0, 1/\rho_D]$ and $(\theta, x) \in \tilde{\Xi}_\beta^{-1}(\Gamma)$ due to Proposition 4.1 (a). However, for big enough b , this expression is certainly smaller than $\exp(9b\delta_1)$. $(\mathcal{A}11)$ and $(\mathcal{A}12)$ can be seen in a similar fashion. Finally, we need that

$$(\mathcal{A}13) \quad |\partial_x^2 \tilde{\xi}_{\beta,\theta}(x)| < \begin{cases} \alpha_c & \text{for } (\theta, x) \in \mathbb{T}^d \times C \\ \alpha_u^2 & \text{for } (\theta, x) \in \Gamma \cap \tilde{\Xi}_\beta^{-1}(\Gamma), \end{cases}$$

which is true due to, in particular, (3.5) and Proposition 4.2.

There are two more assumptions left which deal with the inverse of $\tilde{\xi}_{\beta,\theta}$.

$$(\mathcal{A}14) \quad |\partial_x^2 \tilde{\xi}_{\beta,\theta}^{-1}(x)| < \alpha_e^{-1} \text{ for each } \theta \notin \mathcal{I}_0 + \omega \text{ and } x \in E;$$

$$(\mathcal{A}15) \quad |\partial_\theta \partial_x \tilde{\xi}_{\beta,\theta}^{-1}(x)| < S \alpha_e^{-1} \text{ for each } \theta \notin \mathcal{I}_0 + \omega, x \in E \text{ and } \theta \in \mathbb{S}^{d-1}.$$

Observe that $\tilde{\xi}_{\beta,\theta}^{-1} = \tilde{\xi}_{-\beta,\theta}^-$ (see (1.4) and (1.5)). Hence, we can derive the desired estimates for $\tilde{\xi}_{\beta,\theta}^-$ by means of (3.5) and (3.6) if we replace F_β by F_β^- and ρ by $\rho^- = -\rho$. Under the assumption of $x \in E$ and $\theta \notin \mathcal{I}_{0,\beta} + \rho$, we have that $\tilde{\xi}_\beta^-(t, \theta, x) \in E$ for all $t \in [0, 1/\rho_D]$ and hence $\partial_x \tilde{\xi}_\beta^-(t, \theta, x) \leq \exp(-2b(1 - \exp[-b/(2\rho_D)]) \cdot t)$. Thus, $(\mathcal{A}14)$ follows immediately for large enough b . $(\mathcal{A}15)$ follows directly from the fact that $\partial_\theta F_\beta^-(t\rho^- + \theta, x) = 0$ and hence $\partial_\theta \tilde{\xi}_\beta^-(t, \theta, x) = 0$ for $\theta \notin \mathcal{I}_{0,\beta} + \rho$ and $t \in [0, 1/\rho_D]$.

4.1 Occurrence of a non-smooth bifurcation

We are now in a position to recast Theorem 1.11 by spelling out the definition of $\mathcal{U}_\omega(X)$ in a way adapted to the first return maps $\tilde{\Xi}_\beta$ corresponding to (*).

Given $\alpha, p > 1$ and $K \in \mathbb{N}$, set $q = 1 - 1/K$ and

$$\nu = s - \kappa(\alpha, q) S^2 \alpha^{-(2q^2/p - 5(1-q^2)p)},$$

where $\kappa(\alpha, q)$ is decreasing⁸ in both α and q and s is the lower bound in $(\mathcal{A}7)$. Our reformulation of Theorem 1.11 reads as follows.

Theorem 4.4 (cf. [14, Theorem 4.18] and [13, Theorem 4.2.15]). *Suppose $\omega \in \mathbb{T}^d$ is Diophantine of type (\mathcal{C}', η) , $X \subseteq \mathbb{R}$ is some non-degenerate interval and $(\tilde{\xi}_\beta)_{\beta \in [0,1]}$ lies in $\mathcal{P}_\omega(X)$ and satisfies $(\mathcal{A}1)$ – $(\mathcal{A}15)$. Let there be $p \geq \sqrt{2}$ and $\alpha > 1$ with*

$$\alpha_c^{-1} = \alpha_e \geq \alpha^{2/p}, \quad \alpha_l^{-1} = \alpha_u \leq \alpha^p.$$

Further, assume $3|\mathcal{I}_0| < \mathcal{C}'(2KM)^{-\eta}$ for some integers M not smaller than 2 and K such that $2q^2/p - 5(1 - q^2)p > 0$ and assume $\nu > 0$ as well as $\alpha > \alpha_0$, where $\alpha_0 = \alpha_0(\nu, K, M, p, |C|, |E|, \eta, \mathcal{C}')$. Then there is $\beta_c \in [0, 1]$ such that $\tilde{\xi}_{\beta_c}$ has an SNA and an SNR.

Remark. Here, $|C|$ and $|E|$ denote the length of the intervals of contraction and expansion, respectively. It is important to mention that α_0 can be chosen to be non-increasing in ν and non-decreasing in $|C|$ and $|E|$ for fixed K, M, p, η and \mathcal{C}' .

We want to show that $(\tilde{\Xi}_\beta)_{\beta \in [0,1]}$ verifies the hypothesis of Theorem 4.4 if $R \geq \mathcal{R}$ (for some $\mathcal{R} = \mathcal{R}(|\rho|, \mathcal{C}, \eta)$) and b is large enough. It is straightforward to see that $(\tilde{\Xi}_\beta)_{\beta \in [0,1]} \in \mathcal{P}_\omega(\mathbb{R})$ and that ω is Diophantine (cf. Section 3.2). Now, assume that c and δ_1 are small enough⁹ so that

$$2b(1 - c)(1/\rho_D - \delta_1) - 10b\delta_1 > b(1 + c)/\rho_D.$$

⁸Here, we only need the decreasing behaviour of the constant κ . The interested reader is referred to [13, Lemma 4.2.13] for further details.

⁹In the case of δ_1 , this essentially amounts to assuming small enough R .

Then, setting $\alpha = \exp(b(1+c)/\rho_D)$ and $p = 2$ ensures $\alpha_c^{-1} = \alpha_e \geq \alpha^{2/p}$ and $\alpha_l^{-1} = \alpha_u = \alpha^p$. We have just seen in this section that (A1)–(A15) are verified by $(\tilde{\Xi}_\beta)_{\beta \in [0,1]}$. In fact, observe that (A1)–(A15) still hold when we set the lower bound of the expanding interval E to be $-1 - \varepsilon$ (for some sufficiently small $\varepsilon = \varepsilon(b) > 0$) instead of -1 . Note further that we can choose α as big as we need by assuming large enough b .¹⁰

In Theorem 4.4, we moreover assume that

$$3|I_0| < \mathcal{C}'(2KM)^{-\eta} \quad (4.20)$$

for some positive integers M not smaller than 2 and K such that $q = 1 - 1/K$ satisfies $2q^2/p - 5(1 - q^2)p = q^2 - 10(1 - q^2) > 0$. Observe that—given some $M \in \mathbb{N}_{\geq 2}$ and such K —(4.20) holds true under the assumption of small enough R (independent of b).

Finally, we need

$$v = s - \kappa(\alpha, q) S^2 \alpha^{-(q^2 - 10(1 - q^2))}$$

to be positive. Now, with S as above and $s > \exp(b\delta_2/4)$ (cf. Lemma 4.3), we get

$$v > \exp(b\delta_2/4) - \kappa(\exp(b(1+c)/\rho_D), q) \exp(-b(1+c)[q^2 - 10(1 - q^2)]/\rho_D + 18b\delta_1),$$

which is positive (for sufficiently large b) and increasing in b as long as q is close to 1 and hence, as long as R is small. Altogether, this shows: for big enough R (independent of b and h) and big enough b , $(\tilde{\Xi}_\beta)_{\beta \in [0,1]}$ verifies the assumptions of Theorem 4.4.

Let us fix a Diophantine $\rho \in \mathbb{R}^D$ and only consider families of flows $\hat{\Xi}$ driven by $(t, \theta) \mapsto t \cdot \rho + \theta$ in the following. We define $\mathcal{U}_\rho(X)$ to be the set of all $\hat{F} \in \mathcal{P}(X)$ which generate families $\hat{\Xi}$ with $\hat{\Xi} \in \mathcal{U}_\omega(X)$, that is, $\hat{\Xi}$ verifies the assumptions of Theorem 4.4. From the above, we see that there exists $\hat{F} \in \mathcal{U}_\rho(X)$ such that any C^2 -small perturbation of $\hat{\Xi}$ still lies in $\mathcal{U}_\omega(X)$. Since C^2 -small changes of \hat{F} (recall that we actually consider the modified vector field [cf. Section 3.2]) result in C^2 -small changes of $\hat{\Xi}$ [45, §12 Satz VI], this proves that C^2 -small changes of $\hat{F} \in \mathcal{U}_\rho(X)$ still lie in $\mathcal{U}_\rho(X)$. In other words, Theorem 2.2 holds true for $X = \mathbb{R}$. In fact, a straightforward adaption of (*) immediately yields Theorem 2.2 for arbitrary non-degenerate intervals $X \subseteq \mathbb{R}$.

4.2 Geometry of the invariant graphs

To close the discussion of the continuous time case, let us see how Theorem 1.13 extends to Theorem 2.3. We denote the boundary graphs of the maximal invariant set $\tilde{\Lambda}_{\beta_c}$ of $\tilde{\Xi}_{\beta_c}$ by $\psi_{\tilde{\Lambda}_{\beta_c}}^\pm$ and those of the maximal invariant set Λ_{β_c} of Ξ_{β_c} by $\phi_{\Lambda_{\beta_c}}^\pm$. Notice that

$$\Lambda_{\beta_c} = \Xi_{\beta_c}([0, 1/\rho_D] \times \tilde{\Lambda}_{\beta_c}) \quad \text{and} \quad \Phi_{\Lambda_{\beta_c}}^\pm = \Xi_{\beta_c}([0, 1/\rho_D] \times \Psi_{\tilde{\Lambda}_{\beta_c}}^\pm). \quad (4.21)$$

We restrict to $\phi_{\Lambda_{\beta_c}}^+$ since $\phi_{\Lambda_{\beta_c}}^-$ can be dealt with similarly. The uniqueness of the semi-continuous representatives of $\phi_{\Lambda_{\beta_c}}^+$ and item (iii) are immediate. Since $\phi_{\Lambda_{\beta_c}}^+$ and $\phi_{\Lambda_{\beta_c}}^-$ are $\text{Leb}_{\mathbb{T}^D}$ -almost surely distinct, Lemma 1.8 gives $D_B(\Phi_{\Lambda_{\beta_c}}^+) = D_B(\overline{\Phi_{\Lambda_{\beta_c}}^+}) = D + 1$.¹¹

For the remaining properties, note that the remark in Section 1.4 implies that we just have to show that a statement similar to Proposition 1.12 holds true in the continuous time case. Let $\tilde{\Omega}_j$ be as in

¹⁰Which amounts to assuming big K and hence, small R .

¹¹We may view $\overline{\Phi_{\Lambda_{\beta_c}}^+}$ as a subset of \mathbb{R}^{D+1} .

Proposition 1.12 and set $\Omega_j = [\tilde{\Omega}_j, \tilde{\Omega}_j + \omega]$ where $j \in \mathbb{N}$. Observe that

$$\phi_{\Lambda_{\beta_c}}^+ \upharpoonright_{\Omega_j} : \Omega_j \ni ((\theta, 0) + \theta_D / \rho_D \cdot \rho) \mapsto \Xi_{\beta_c} \left(\theta_D / \rho_D, (\theta, 0), \psi_{\Lambda_{\beta_c}}^+ (\theta) \right)$$

is Lipschitz continuous. Finally, consider $\Omega_\infty = \mathbb{T}^D \setminus \bigcup_{j \in \mathbb{N}} \Omega_j \subseteq [\tilde{\Omega}_\infty, \tilde{\Omega}_\infty + \omega]$. Due to Lemma 1.6, Theorem 1.7, and Proposition 1.12

$$D_H([\tilde{\Omega}_\infty, \tilde{\Omega}_\infty + \omega]) = D_H(\tilde{\Omega}_\infty \times [0, 1/\rho_D]) \leq d.$$

Due to the monotonicity of the Hausdorff dimension, we hence get $D_H(\Omega_\infty) \leq D_H([\tilde{\Omega}_\infty, \tilde{\Omega}_\infty + \omega]) \leq d$ and thus we have an analogue to Proposition 1.12. This finishes the proof of Theorem 2.3.

References

- [1] L. Ambrosio and B. Kirchheim. Rectifiable sets in metric and Banach spaces. *Math. Ann.*, 318(3):527–555, 2000.
- [2] V. Anagnostopoulou and T. Jäger. Nonautonomous saddle-node bifurcations: random and deterministic forcing. *J. Differential Equations*, 253(2):379–399, 2012.
- [3] L. Arnold. *Random dynamical systems*. Springer, Berlin Heidelberg, 1998.
- [4] M. Arrigoni and A. Steiner. Logistisches Wachstum in fluktuierender Umwelt. *J. Math. Biol.*, 21(3):237–241, 1985.
- [5] M. Bardi. An equation of growth of a single species with realistic dependence on crowding and seasonal factors. *J. Math. Biol.*, 17(1):33–43, 1983.
- [6] K. Bjerklov. Dynamics of the quasi-periodic Schrödinger cocycle at the lowest energy in the spectrum. *Comm. Math. Phys.*, 272(2):397–442, 2007.
- [7] K. Bjerklov. Positive Lyapunov exponent and minimality for the continuous 1-d quasi-periodic Schrödinger equation with two basic frequencies. *Ann. Henri Poincaré*, 8(4):687–730, 2007.
- [8] F. Brauer and D. A. Sánchez. Periodic environments and periodic harvesting. *Nat. Resour. Model.*, 16:233–244, 2003.
- [9] E. Braverman and R. Mamdani. Continuous versus pulse harvesting for population models in constant and variable environment. *J. Math. Biol.*, 57(3):413–434, 2008.
- [10] C. Castilho and P. D. N. Srinivasu. Bio-economics of a renewable resource in a seasonally varying environment. *Math. Biosci.*, 205(1):1–18, 2007.
- [11] J. M. Cushing. Oscillatory population growth in periodic environments. *Theor. Popul. Biol.*, 30(3):289–308, 1986.
- [12] K. J. Falconer. *Fractal geometry - mathematical foundations and applications*. Wiley, Chichester, 2nd edition, 2003.
- [13] G. Fuhrmann. *Strange attractors of forced one-dimensional systems: existence and geometry*. PhD thesis, Friedrich-Schiller-Universität Jena, year=2015, url=<http://www.db-thueringen.de/servlets/DerivateServlet/Derivate-32462/Diss/FUHRMANN.pdf>.

-
- [14] G. Fuhrmann. Non-smooth saddle-node bifurcations I: existence of an SNA. *Ergodic Theory Dynam. Systems*, FirstView:1–26, 12 2014.
- [15] G. Fuhrmann, M. Gröger, and T. Jäger. Non-smooth saddle-node bifurcations II: dimensions of strange attractors. *arXiv:1412.6054v2*, 2014.
- [16] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke. Strange attractors that are not chaotic. *Phys. D*, 13:261–268, August 1984.
- [17] M. Gröger and T. Jäger. Dimensions of attractors in pinched skew products. *Comm. Math. Phys.*, 320(1):101–119, 2013.
- [18] J. M. Gushing and S. M. Henson. Global dynamics of some periodically forced, monotone difference equations. *J. Difference Equ. Appl.*, 7(6):859–872, 2001.
- [19] P. Hartman. *Ordinary differential equations*. Wiley, New York London Sydney, 1st edition, 1964.
- [20] M. R. Herman. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d’un théorème d’Arnold et de Moser sur le tore de dimension 2. *Comment. Math. Helv.*, 58(1):453–502, 1983.
- [21] J. D. Howroyd. On dimension and on the existence of sets of finite positive Hausdorff measure. *Proc. Lond. Math. Soc.*, s3-70(3):581–604, 1995.
- [22] J. D. Howroyd. On Hausdorff and packing dimension of product spaces. *Math. Proc. Cambridge Philos. Soc.*, 119(04):715–727, 1996.
- [23] T. Jäger. Quasiperiodically forced interval maps with negative Schwarzian derivative. *Nonlinearity*, 16(4):1239–1255, 2003.
- [24] T. Jäger. On the structure of strange non-chaotic attractors in pinched skew products. *Ergodic Theory Dynam. Systems*, pages 493–510, 2007.
- [25] T. Jäger. The creation of strange non-chaotic attractors in non-smooth saddle-node bifurcations. *Mem. Amer. Math. Soc.*, 201(945):1–106, 2009.
- [26] T. Jäger. Strange non-chaotic attractors in quasiperiodically forced circle maps. *Comm. Math. Phys.*, 289:253–289, 2009. 10.1007/s00220-009-0753-0.
- [27] D. A. Jillson. Insect populations respond to fluctuating environments. *Nature*, 288:699–700, 1980.
- [28] R. A. Johnson. On almost-periodic linear differential systems of Millionščikov and Vinograd. *J. Math. Anal. Appl.*, 85(2):452–460, 1982.
- [29] À. Jorba, J. C. Tatjer, C. Núñez, and R. Obaya. Old and new results on strange nonchaotic attractors. *Int. J. Bifurc. Chaos Appl. Sci. Eng.*, 17(11):3895–3928, 2007.
- [30] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.
- [31] G. Keller. A note on strange nonchaotic attractors. *Fund. Math.*, 151(2):139–148, 1996.

-
- [32] V. M. Koltyzhenkov. Irregular second-order differential equations with almost-periodic coefficients. *Math. Notes*, 41(2):116–118, 1987.
 - [33] A. V. Lipsnitskii. Millionshchikov’s solution of Erugin’s problem. *Differ. Equ.*, 36(12):1770–1776, 2000.
 - [34] P. Liu, J. Shi, and Y. Wang. Periodic solutions of a logistic type population model with harvesting. *J. Math. Anal. Appl.*, 369(2):730–735, 2010.
 - [35] V. M. Millionščikov. Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients. *Differ. Equ.*, 4(3):391–396, 1968.
 - [36] C. Núñez and R. Obaya. A non-autonomous bifurcation theory for deterministic scalar differential equations. *Discrete Contin. Dyn. Syst. Ser. B*, 9(3/4, May):701–730, 2008.
 - [37] S. Oruganti, J. Shi, and R. Shivaji. Diffusive logistic equation with constant yield harvesting, I: Steady states. *Trans. Amer. Math. Soc.*, 354(9):3601–3619, 9 2002.
 - [38] S. Rosenblat. Population models in a periodically fluctuating environment. *J. Math. Biol.*, 9(1):23–36, 1980.
 - [39] D. M. Sonechkin and N. N. Ivachtchenko. On the role of quasiperiodic forcing in the interannual and interdecadal climate variations. *CLIVAR exchanges*, 6:5–6, 2001.
 - [40] R. B. Spies. Chapter 3 - agents of ecosystem change. In R. B. Spies, editor, *Long-term ecological change in the northern Gulf of Alaska*, pages 171–257. Elsevier, Amsterdam, 2007.
 - [41] J. Stark. Invariant graphs for forced systems. *Phys. D*, 109(1–2):163–179, 1997. Proceedings of the Workshop on Physics and Dynamics between Chaos, Order, and Noise.
 - [42] J. Stark. Transitive sets for quasi-periodically forced monotone maps. *Dyn. Syst.*, 18(4):351–364, 2003.
 - [43] R. R. Vance and E. A. Coddington. A nonautonomous model of population growth. *J. Math. Biol.*, 27(5):491–506, 1989.
 - [44] R. E. Vinograd. A problem suggested by N.R. Erugin. *Differ. Uravn.*, 11(4):632–638, 1975.
 - [45] W. Walter. *Gewöhnliche Differentialgleichungen – Eine Einführung*. Springer-Verlag, Berlin Heidelberg New York, 2nd edition, 1976.
 - [46] C. Xu, M. S. Boyce, and D. J. Daley. Harvesting in seasonal environments. *J. Math. Biol.*, 50(6):663–682, 2005.
 - [47] L.-S. Young. Dimension, entropy and Lyapunov exponents. *Ergodic Theory Dynam. Systems*, 2:109–124, 1982.
 - [48] L.-S. Young. Lyapunov exponents for some quasi-periodic cocycles. *Ergodic Theory Dynam. Systems*, 17:483–504, 4 1997.
 - [49] O. Zindulka. Hentschel-Procaccia spectra in separable metric spaces, 2002. Real Analysis Exchange, Summer Symposium in Real Analysis XXVI:115–119.